ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES*

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1. **Introduction.** A series $\sum u_n$ is said to be absolutely summable by a method $\mathfrak{a}$ defined by a matrix a_{mn} if

$$\sum_{m=1}^{\infty} \left| S_m(\mathfrak{a}, u) - S_{m-1}(\mathfrak{a}, u) \right| < \infty,$$

where

$$S_m(\mathfrak{a}, u) = \sum_{n=0}^{\infty} a_{mn} u_n.$$

Similarly a series is said to be absolutely summable $|A|$ if

$$S(r, u) = \sum_{n=0}^{\infty} u_n r^n \subset BV \quad \text{on} \quad (0, 1).$$

It is known that if $\sum u_n$ is absolutely summable $|C_\alpha|$ for some $\alpha > 0$, then it is absolutely summable $|A|$. There are, however, series absolutely summable $|A|$ but not $|C_\alpha|$ for any α whatever. We intend to give here an example of a Fourier series with that property.

Bosanquet† has proved that, if the Fourier series of $f(x)$ is absolutely summable $|C_\alpha|$, then the function

$$\phi_\beta(f, t) = \beta t^{-\beta} \int_0^t \{f(x+\tau) + f(x-\tau) - 2f(x)\} (t-\tau)^{\beta-1} d\tau$$

is of bounded variation on $(0, \pi)$ for $\beta > \alpha$; and conversely, if $\phi_\alpha(t)$ is of bounded variation, the Fourier series of $f(x)$ is absolutely summable $|C_\beta|$, $(\beta > \alpha + 1)$.

2. **Preliminary definitions.** Let α_{nk}, β_{nk} be defined for $n = 1, 2, \cdots$, $k = 1, 2, \cdots, n$, by

$$(1) \qquad \alpha_{nk} = 2^{-k-n-n/(k-1/2)}, \qquad \beta_{nk} = 2^{-n} - 2^{-n-n/(k-1/2)}.$$

Then, since $k \leqq n$, we have

$$\beta_{nk} > 2^{-n-1}.$$

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† L. S. Bosanquet, *The absolute Cesàro summability of Fourier series*, Proceedings of the London Mathematical Society, vol. 41 (1936), pp. 517–528.