TRANSFORMATION OF BASES FOR RELATIVE LINEAR SETS*

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The definitions of linear independence, dependence, and extension of sets of vectors relative to a matrix which are used in this paper were recently introduced by M. H. Ingraham,‡ and are not repeated here. The purpose of the present paper is to study the structure of basal elements of a linear extension relative to a matrix. There are developed necessary and sufficient conditions which the elements of a matrix of a transformation must satisfy so that one set of basal elements of a vector space can be transformed into another basal set.

It is assumed throughout this paper that the elements of the matrices and the vectors, as well as the coefficients of the polynomials, are in a field $\mathfrak{F}$. The following theorem is used but stated without proof:§

THEOREM 1. *If $\xi_1, \xi_2, \cdots, \xi_k$ and $\eta_1, \eta_2, \cdots, \eta_l$ are two sets of vectors ($n\times 1$ matrices), each of which is linearly independent relative to an $n\times n$ matrix M, such that $L_M(\xi_1, \xi_2, \cdots, \xi_k)=L_M(\eta_1, \eta_2, \cdots, \eta_l)$, if h_{1i} and h_{2i} are, respectively, the minimum polynomials associated with ξ_i and η_i relative to M, and if g is an irreducible polynomial and t any positive integer, then the number of polynomials h_{1i} divisible by g^t is equal to the number of h_{2i} divisible by g^t.*

This is essentially the only restriction on the polynomials h_{1i} and h_{2i}.

THEOREM 2. *If $\xi_1, \xi_2, \cdots, \xi_k$ is a proper base relative to a matrix M for the space $L_M(\xi_1, \xi_2, \cdots, \xi_k)$, if the minimum polynomial associated with ξ_i relative to M is g^{t_i}, a power of an irreducible polynomial g, and if $\eta_i=\sum_{j=1}^{k} f_{ij}(M)\xi_j$, $(i=1, 2, \cdots, k)$, where the f_{ij} are polynomials with coefficients in the field $\mathfrak{F}$, necessary and sufficient conditions that the η_i form a proper base relative to M are that the set of polynomials minimally associated with the η_i is exactly the set g^{t_i} in some order, and*

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‡ M. H. Ingraham and M. C. Wolf, *Relative linear sets and similarity of matrices whose elements belong to a division algebra*, Transactions of this Society, vol. 42 (1937), pp. 16–31.

§ M. H. Ingraham and M. C. Wolf, loc. cit.