The stronger result of 3.3 shows that $l(r, \theta; \phi\bar{\phi})$ is of class PL; whence

$$[l(r, \theta; \phi\bar{\phi})]^2 \Delta \log l(r, \theta; \phi\bar{\phi})$$

$$(17)\qquad = \sum_{g,h,j,k=1}^{\infty} \frac{a_g \bar{a}_h a_j \bar{a}_k (g-h)^2 (j-k)^2}{(g+h+1)(j+k+1)(g+k+1)(h+j+1)} \cdot r^{g+h+j+k} e^{i(g-h+j-k)} \geqq 0.$$

On the other hand, it can be shown directly, by an extension of the above identities, that (17) is positive definite.

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SOME ITERATED INTEGRALS IN THE FRACTIONAL CALCULUS

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1. Introduction. A considerable amount of attention has been devoted to integrals of fractional order, both in regard to their applications and to the conditions for their existence.* We shall denote the fractional integral of order α by

$$(1)\qquad {}_TI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_T^t (t-v)^{\alpha-1} f(v)\,dv, \qquad \alpha > 0,\ t > T,$$

and it is the purpose of this paper to give some formulas which may be of use in manipulating these integrals. We shall prove that under certain conditions the following relations hold:

$$(2)\qquad \int_T^{\infty} \frac{{}_TI_t^{\alpha} f(t)}{t^{k+\alpha}}\,dt = \frac{\Gamma(k)}{\Gamma(k+\alpha)} \int_T^{\infty} \frac{f(t)}{t^k}\,dt, \qquad \alpha > 0,$$

$$(3)\qquad \int_T^{\infty} e^{-kt}\,{}_TI_t^{\alpha} f(t)\,dt = k^{-\alpha} \int_T^{\infty} e^{-kt} f(t)\,dt, \qquad \alpha > 0,$$

$$(4)\qquad \int_T^{\infty} \cos kt\,{}_TI_t^{\alpha} f(t)\,dt = k^{-\alpha} \int_T^{\infty} \cos (kt + \pi\alpha/2) f(t)\,dt, \quad 0 < \alpha < 1,$$

and (4) holds when cosine is replaced by sine. As an application we

* A bibliography is given by H. T. Davis, *Application of fractional operators to functional equations*, American Journal of Mathematics, vol. 49 (1927), pp. 123–142. See also J. D. Tamarkin, *On integrable solutions of Abel's integral equation*, Annals of Mathematics, (2), vol. 31 (1930), pp. 219–229.