A RELATIVE OF THE LEMMA OF SCHWARZ*

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1. **Introduction.** Let $w=f(z)$, where $z=u+iv$, be analytic for $|z|<1$, and let

$$d(r, \theta; f') = |f(re^{i\theta}) - f(0)| = \left|\int_0^r f'(\rho e^{i\theta})d\rho\right|,$$

so that $d(r, \theta; f')$ is the length of the segment on the w-plane between the image of the point $z=0$ and the image of the point $z=re^{i\theta}$.

The lemma of Schwarz is the following:

THEOREM 1. *Let $w=f(z)$ be analytic for $|z|<1$. If*

$$d(r, \theta; f') \leqq 1$$

for all (r, θ) with $r<1$, then

$$d(r, \theta; f') \leqq r \tag{1}$$

and

$$|f'(0)| \leqq 1. \tag{2}$$

The sign of equality holds in (1) (*for $r\neq 0$*) *and in* (2), *if and only if $|f'(z)| \equiv 1$; that is, if and only if the transformation $w=f(z)$ is a rigid motion.*

If the (real) function $g(z)$ is subharmonic for $|z|<1$, then the Lebesgue integral

$$l(r, \theta; g) = \int_0^r g(\rho e^{i\theta})d\rho$$

actually exists. We shall prove the following theorem:

THEOREM 2. *Let $g(z)$ be subharmonic for $|z|<1$. If*

$$l(r, \theta; g) \leqq 1 \tag{3}$$

for all (r, θ) with $r<1$, then

$$l(r, \theta; g) \leqq r \tag{4}$$

and

$$g(0) \leqq 1. \tag{5}$$

* Presented to the Society, December 28, 1937.