A SUFFICIENCY PROOF FOR ISOPERIMETRIC PROBLEMS IN THE CALCULUS OF VARIATIONS

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The purpose of the present paper is to show that the sufficiency theorems for a strong relative minimum for isoperimetric problems can be obtained from those for simple integral problems with only a little additional argument. The method here used is a simple extension of one used by Birkhoff and Hestenes* for a special isoperimetric problem. Heretofore sufficiency theorems of this type have been obtained from those for a restricted relative minimum by the application of the theorem of Lindeberg. Sufficiency theorems for a restricted relative minimum can be obtained either from the theories of the problems of Lagrange and Bolza or by an argument analogous to that used for simple integrals.

The problem to be considered is that of minimizing an integral

$$J = \int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

in a class of admissible arcs

$$y_i(x), \qquad x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n, \tag{1}$$

joining two fixed points 1 and 2 and satisfying a set of isoperimetric conditions

$$J_\alpha = \int_{x_1}^{x_2} f_\alpha(x, y, y')dx = l_\alpha, \qquad \alpha = 1, \cdots, m, \tag{2}$$

where the l's are constants. It will be assumed that the functions f, f_α are defined and have continuous derivatives of the first three orders in a region $\mathcal{R}$ of points (x, y, y'). The points of $\mathcal{R}$ will be called *admissible.* A continuous arc (1) that can be divided into a finite number of subarcs on each of which it has continuous derivatives will be called *admissible* if its elements (x, y, y') are all admissible.

Associated with the problem is an integral of the form

$$I_\lambda = \int_{x_1}^{x_2} F(x, y, y', \lambda)dx,$$

* *Natural isoperimetric conditions in the calculus of variations,* Duke Mathematical Journal, vol. 1 (1935), pp. 251–258. The method used in this paper was suggested by Professor Birkhoff.