CARTAN ON GROUPS AND DIFFERENTIAL GEOMETRY

La Théorie des Groupes Finis et Continus et la Géométrie Différentielle traitées par la Méthode du Repère Mobile. By Élie Cartan. (Cahiers Scientifiques, no. 18.) Paris, Gauthier-Villars, 1937. 6+269 pp.

This book, which originated from a course of lectures given in 1931–1932 at the Sorbonne, covers in a somewhat more explicit form essentially the same material as no. 194 (1935) of the Actualités Scientifiques et Industrielles (see the review, this Bulletin, vol. 41 (1935), p. 774). By means of the method of the repère mobile the author studies arbitrary manifolds M_λ in a Klein space R whose geometry is described by its group of automorphisms. The chief aim of this review shall be to bring out the axiomatic foundations of the theory.

Coördinatization of a space R consists in a one-to-one mapping of the points A of R upon a manifold Σ of (numerical) symbols x serving as coördinates. In a Klein space such coördinatization is possible only with respect to a frame of reference, or briefly frame, $\mathfrak{f}$. An abstract group G and a realization of it by means of one-to-one transformations of Σ are supposed to be given. We thus deal with four kinds of objects: points A, symbols x, frames $\mathfrak{f}$, and group elements s, their mutual relation being established by two axioms (a) and (b):

(a) Any pair of frames $\mathfrak{f}, \mathfrak{f}'$ determines a group element $s=(\mathfrak{f}\rightarrow\mathfrak{f}')$ called the transition from $\mathfrak{f}$ to $\mathfrak{f}'$. Vice versa, any element s of G carries a given frame $\mathfrak{f}$ into a uniquely determined frame $\mathfrak{f}'$ such that $s=(\mathfrak{f}\rightarrow\mathfrak{f}')$. Succession of transitions $\mathfrak{f}\rightarrow\mathfrak{f}'\rightarrow\mathfrak{f}''$ corresponds to the composition of group elements:

$$s = (\mathfrak{f} \rightarrow \mathfrak{f}'),\quad t = (\mathfrak{f}' \rightarrow \mathfrak{f}'')\quad \text{imply}\quad ts = (\mathfrak{f} \rightarrow \mathfrak{f}'').$$

(The identical element is $\mathfrak{f}\rightarrow\mathfrak{f}$, and $\mathfrak{f}'\rightarrow\mathfrak{f}$ is the inverse of $\mathfrak{f}\rightarrow\mathfrak{f}'$.)

(b) With respect to a given frame $\mathfrak{f}$ each point A determines a symbol $x=(A, \mathfrak{f})$ as its coördinate, thus setting up a one-to-one correspondence $A\rightleftarrows x$ between R and Σ. The coördinate $x'=(A, \mathfrak{f}')$ of A in another frame $\mathfrak{f}'$ arises from x by the transformation associated with the group element $s=(\mathfrak{f}\rightarrow\mathfrak{f}')$ in the given realization.

Consequences: $\mathfrak{f}$, $\mathfrak{f}^*$ being any two frames, the equation $(A^*, \mathfrak{f}^*)=(A, \mathfrak{f})$ defines a one-to-one mapping $A\rightarrow A^*$ of R upon itself, the *space automorphism* $\{\mathfrak{f}, \mathfrak{f}^*\}$. If a group element t changes $\mathfrak{f}$, $\mathfrak{f}^*$ into $\mathfrak{g}, \mathfrak{g}^*$, one evidently has at the same time $(A^*, \mathfrak{g}^*)=(A, \mathfrak{g})$. Hence the space automorphisms $\{\mathfrak{f}, \mathfrak{f}^*\}$ form a group isomorphic with G; but their isomorphic mapping onto the elements of G is fixed except for an arbitrary inner automorphism of the group G. Figures in R which arise from each other by space automorphisms are considered *equal.*

We are concerned with λ-dimensional parametrized manifolds

$$M_\lambda:\quad x = x(t_1, \cdots, t_\lambda),$$

where t_α are real parameters. Let us employ for a moment ordinary real coördinates in R, $x=(x_1, \cdots, x_n)$. At a given point $A=(t_1, \cdots, t_\lambda)$ of M_λ the functions $x_i(t_1, \cdots, t_\lambda)$ and their derivatives up to a given order p constitute a contact element of order p, or briefly a p-spread. We obtain a succession of such spreads of orders $p=0, 1, 2, \cdots$, each of which is contained in the following. The central problem consists in deciding when two parametrized manifolds M_λ, M_λ' are equal. In the analytic case one may ask instead under what circumstances two given p-spreads are equal,