THE TRANSFORMS OF FUCHSIAN GROUPS

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This paper will deal with Fuchsian linear transformations that have the unit circle

$$Q_0(z) \equiv z\bar{z} - 1 = 0$$

as principal circle. The transformations have the form

$$T(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \qquad a\bar{a} - c\bar{c} = 1. \tag{1}$$

We shall develop several theorems concerning the relative size of the isometric circle of any transformation T of the group and of the isometric circle of

$$\begin{aligned} S(z) &= GTG^{-1}(z) \\ &= \frac{(-\alpha\bar{\alpha}a + \alpha\nu\bar{c} - \bar{\alpha}\bar{\nu}c + \nu\bar{\nu}\bar{a})z + \alpha\bar{\nu}a - \alpha^2\bar{c} + \bar{\nu}^2c - \alpha\bar{\nu}\bar{a}}{(-\bar{\alpha}\nu a + \nu^2\bar{c} - \bar{\alpha}^2c + \bar{\alpha}\nu\bar{a})z + \nu\bar{\nu}a - \alpha\nu\bar{c} + \bar{\alpha}\bar{\nu}c - \alpha\bar{\alpha}\bar{a}} \\ &= \frac{Az + \bar{C}}{Cz + \bar{A}}, \qquad A\bar{A} - C\bar{C} = 1, \end{aligned} \tag{2}$$

in which

$$G(z) = \frac{\alpha z + \bar{\nu}}{\nu z + \bar{\alpha}}, \qquad \alpha\bar{\alpha} - \nu\bar{\nu} = 1,$$

is variable and T fixed.

The concept of the isometric circle of a linear transformation was introduced in 1926 by L. R. Ford. Because of his work with it, the isometric circle now plays a fundamental part in work with groups of linear transformations. By definition* the isometric circle of a linear transformation is the complete locus of points in the neighborhood of which lengths and areas are unaltered in magnitude by the transformation. For convenience we shall sometimes use the designation $Q(z)$ for a line or circle $Q(z) = 0$.

THEOREM 1. *A necessary and sufficient condition that the radii r_s and r_t of the isometric circles of S and T be equal is that the center w of the isometric circle of the transformation G be either on the line $Q_1(z)$ through the center g_t of the isometric circle of T and the center g_t' of the isometric circle of T^{-1} or on the inverse $Q_2(z)$ of $Q_1(z)$ in $Q_0(z)$.*

* L. R. Ford, *Automorphic Functions*, p. 25.