REMARKS ON THE CLASSICAL INVERSION FORMULA FOR THE LAPLACE INTEGRAL

D. V. WIDDER AND N. WIENER

If a function $f(s) = f(\sigma + i\tau)$ is defined for $\sigma > 0$ by the Laplace integral

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt, \tag{1}$$

then the classical inversion formula is

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{st}ds, \qquad c > 0, t > 0. \tag{2}$$

Conditions for the validity of this formula have frequently been discussed. However, the authors know of no adequate treatment* of the case when $\phi(t)$ belongs to L^2 in $(0, \infty)$:

$$\int_0^\infty |\phi(t)|^2 dt < \infty. \tag{3}$$

We employ here the usual notation,

$$\underset{a\to\infty}{\text{l.i.m.}}\ \phi_a(t) = \phi(t),$$

to mean that $\phi_a(t)$ and $\phi(t)$ belong to L^2 in $(-\infty, \infty)$ and that

$$\lim_{a\to\infty}\int_{-\infty}^\infty |\phi_a(t) - \phi(t)|^2 dt = 0.$$

It is clear first that if (3) holds then (1) converges absolutely for $\sigma > 0$, since

$$|f(\sigma + i\tau)|^2 = \left|\int_0^\infty e^{-st}\phi(t)dt\right|^2 \leqq \int_0^\infty e^{-2\sigma t}dt \int_0^\infty |\phi(t)|^2 dt.$$

Moreover, by the Plancherel theorem regarding Fourier transforms,

$$\underset{a\to\infty}{\text{l.i.m.}} \int_0^a e^{-i\tau t}\phi(t)dt$$

* But compare G. Doetsch, *Bedingungen für die Darstellbarkeit einer Funktion als Laplace Integral und eine Umkehrformel für die Laplace-Transformation*, Mathematische Zeitschrift, vol. 42 (1936), p. 272, Theorem 1.