DIVISIBILITY OF GENERALIZED FACTORIALS*

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1. **Introduction.** Two different types of expression were obtained by A. M. Legendre† for H, the index of the highest power of the prime p dividing $n!$:

$$H = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots, \tag{1}$$

$$H = \frac{n-s}{p-1}, \tag{2}$$

where $[a/b]$ denotes the largest integer less than or equal to a/b, and s is the sum of the digits of n to the base p. R. D. Carmichael‡ considered the more general problem of determining H for $\prod_{x=0}^{n-1}(xa+c)$, where a and c are relatively prime positive integers and $a \not\equiv 0 \pmod{p}$. He obtained expressions of type (1) and upper and lower bounds for H. In the present paper a correction is made in the upper bound, new expressions for H of types (1) and (2) are derived, and the results are extended to products where a and c are any positive integers.

2. **Discussion of previous results.** Carmichael used the following method: Set $c_0 = c$, and let i_r be the smallest value of $x \geqq 0$ such that $xa + c_{r-1} \equiv 0 \pmod{p}$, the quotient being c_r. Then $i_r \leqq p-1$. Let $e_0 = n-1$, $e_r = [(e_{r-1} - i_r)/p]$, $(r>0)$. If $\prod_{x=0}^{n-1}(xa+c_0)$ is divisible by p, it has e_1+1 factors of the form $(mp+i_1)a+c_0$, $(0 \leqq m \leqq [(e_0-i_1)/p])$, each divisible by p. The product of the quotients is $\prod_{x=0}^{e_1}(xa+c_1)$. If this product is divisible by p, it has e_2+1 factors of the form $(mp+i_2)a+c_1$, $(0 \leqq m \leqq [(e_1-i_2)/p])$, each divisible by p. Hence e_2+1 factors of $\prod_{x=0}^{n-1}(xa+c_0)$ are divisible by p^2. If the product of the quotients $\prod_{x=0}^{e_2}(xa+c_2)$ is divisible by p, e_3+1 factors of $\prod_{x=0}^{n-1}(xa+c_0)$ are divisible by p^3. Continue in this manner until a product $\prod_{x=0}^{e_t}(xa+c_t)$ is obtained which is not divisible by p. Then e_t+1 factors of the original product are divisible by p^t and no factors by p^{t+1}. Hence

$$H = \sum_{r=1}^{t} (e_r + 1). \tag{3}$$

* Presented to the Society, April 10, 1936. By a generalized factorial we mean a product of integers forming an arithmetic progression.

† *Théorie des Nombres*, 2d edition, 1808, p. 8.

‡ This Bulletin, vol. 15 (1908–1909), pp. 217–221.