POLYGONAL VARIATIONS*

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So-called direct methods in the calculus of variations, such as those of Tonelli involving lower semi-continuity, sometimes insure the existence of an arc minimizing an integral $J=\int f dx$ in cases where many of the partial derivatives of the integrand function f which occur in the usual theory do not exist. Examples are integrals of the form $\int g(x, y)(1+y'^2)^{1/2}dx$, or in the parametric case $\int g(x, y)(x'^2+y'^2)^{1/2}dt$, where $g(x, y)$ is merely continuous and positive. When a minimizing arc is known to exist, necessary conditions assume greater importance. It seems desirable, therefore, to have methods of deriving the familiar necessary conditions of Weierstrass, Euler, and Legendre, while making as few assumptions as possible concerning the existence of partial derivatives of the integrand f. In this paper it is shown that by using the method of polygonal variations the Weierstrass necessary condition and a generalization of the Euler equation can be derived under the assumption of the existence and continuity of the partial derivative $f_{y'}$ only. A slightly generalized form of the Legendre condition can be proved, with the assumption only of the existence of the generalized second partial derivative $f_{y'y'}$. The method involves giving the dependent variable or variables variations whose graphs are polygonal lines of proper shape, depending on a parameter ϵ, and then evaluating the derivative $J'(\epsilon)$ when $\epsilon=0$. Since the method generalizes easily in the usual way to the case of more than one dependent variable and also to the parametric problem, the discussion will be given here only for the simplest case of a non-parametric problem with one dependent variable.

1. **The Weierstrass necessary condition.** Some of the methods commonly used to derive the Weierstrass necessary condition make use of such complicated notions as fields of extremals and Hilbert's invariant integral, and most of them use the Euler equation, which involves the partial derivative f_y. L. M. Graves however, has given a proof (American Mathematical Monthly, vol. 41 (1934), pp. 502–504) which is independent of the Euler equation and requires only the existence and continuity of $f_{y'}$. The following proof by means of polygonal variations is made under the same weak assumptions.

In all that follows the integrand $f(x, y, y')$ is assumed to be con-

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