THE TOPOLOGY OF TRANSFORMATION GROUPS*

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1. Stated in general terms, the problem which I wish to consider is the following: A group G of transformations operates in a space S; what relations must exist between the topology of this situation on the one hand, and its group theoretical properties, on the other? This is of course a rather vague question and I shall not attempt to describe all the recent results which could be considered as being relevant. I hope rather to illustrate certain special phases of the problem by means of examples. I shall then consider in some detail one special case where certain conclusions can be drawn which may possibly be of interest in algebraic geometry. I refer particularly to a generalization to higher dimensions of Harnack's theorem concerning the number of real branches which a real algebraic curve may possess.

Continuous groups

2. Let us begin by supposing that G is a continuous r-parameter group. The elements of G may then be thought of as points of a space which has locally the character of a euclidean r-space. Products and quotients of two elements are to vary continuously with those elements. What restrictions does the fact that G is a continuous group place upon the *space* G? It is very easy to see that G cannot be of arbitrary topological structure. For suppose that a is a fixed element different from the identity, and that x is an arbitrary element; then the transformation T_a: $x \rightarrow ax$, is a homeomorphism of G with itself, and no x remains fixed. Now if we assume that G is connected, a can be made to describe a continuous path toward the identity element, and T_a then undergoes a continuous modification into the identical transformation. That is, T_a belongs to the "class of the identity." But in many spaces (a sphere for example) every transformation which belongs to the class of the identity must leave at least one point fixed. Thus no two-parameter group can be topologically equivalent to a sphere.† It is just as easy to show that G must be

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† A necessary condition that a compact manifold admit transformations of the class of the identity without fixed points is that its Euler-Poincaré characteristic vanish. See Lefschetz, *Topology*, American Mathematical Society Colloquium Publications, vol. 12, New York, 1930, pp. 272, 359.