DOETSCH ON THE LAPLACE TRANSFORMATION

Theorie und Anwendung der Laplace-Transformation. By G. Doetsch. (Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 47.) Berlin, Springer, 1937. 8+436 pp.

A function $F(t)$, $(t \geqq 0)$, is defined here as an L-function if it is bounded in every finite interval in $t>0$, is absolutely integrable in $0 \leqq t \leqq T$ for some $T>0$, and has a convergent Laplace integral $\int_0^\infty e^{-st}F(t)dt$ for some real or complex $s=s_0$. The integrals are taken in the Riemann sense. After it has been proved that this integral then converges for every s in the half-plane $R(s)>R(s_0)$, the class of l-functions is defined as the aggregate of all functions $f(s)$ which are Laplace transforms of L-functions: $f(s)=\int_0^\infty e^{-st}F(t)dt=\mathcal{L}\{F\}$. The principal aims of the book are to show how the character of and operations upon functions belonging to one of these classes are reflected upon the corresponding functions of the other, and to demonstrate a wide variety of applications of these results. Functions of other classes and other transformations play only minor rôles here, except for the two-sided Laplace transformation and the Mellin transformation, the study of which is made to depend in a simple way upon that of $\mathcal{L}\{F\}$. After discussing the advantages of using Lebesgue integrals, limit in the mean, and other concepts in modern analysis, the author presents his reasons for using only the older concepts (Chapter 3, §1; 8). Aside from his intention to adapt the book to readers without such modern equipment, he points out that in the applications the local limit rather than the limit in the mean is needed, and that functions appear there which are L-functions but do not belong to the Lebesgue square integrable class. The Laplace transformation of L-functions into l-functions is not a continuous transformation, so difficulties are encountered here which are not present when the functions belong to Hilbert space.

The book is divided into five parts covering, in order, the general theory, its use in finding expansions of certain functions in series, the asymptotic behavior of functions, integral equations, and differential equations. The applications are said to begin with Part II, but much of Part III is again general theory, so less than half the book is devoted to the applications. The appendix contains a useful table of transforms of functions together with their half-planes of convergence.

Part I first presents a few fundamental concepts of function spaces and transformations. The short historical account of the genesis of the Laplace transformation which follows is supplemented later in Chapter 6 by a section on the early history of the Fourier transform, and again by remarks on the more recent sources among the many reference notes in the appendix. The chapters on general theory give an extensive treatment of half-planes of convergence, the regions of regularity, and analytic continuation of l-functions. It is important for the reader to note that $F(t)$ is assumed to be an L-function when the contrary is not stated; otherwise, several theorems in Chapters 4 and 6 will be confusing. A good account of inversion formulas follows; this includes the complex inversion integral and the formulas of Post, a little-known one by Phragmén, series inversion formulas, and others. The final chapter of this part, on the mapping of the fundamental operations, determines the operations in either the L- or l-space which correspond to linear substitution, integration, differentiation, or multiplication of functions in the other.

A single chapter, illustrating the mapping of series developments of particular functions, makes up Part II. Part III begins with Chapter 10, containing thirteen