TORSION OF REGIONS BOUNDED BY CIRCULAR ARCS*

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1. Introduction. N. Muschelišvili discussed recently† some elegant general methods of solution of a class of problems occurring in the theory of elasticity. These methods will bear closer scrutiny as the present authors have convinced themselves by obtaining, with remarkable ease, solutions of St. Venant's torsion problem for some regions bounded by circular arcs. A number of problems in harmonic and biharmonic analysis, recently discussed by other investigators, can be treated with greater simplicity by these methods.

Muschelišvili makes use of the fact that a function of a complex variable ζ, analytic for $|\zeta| < 1$, whose real part on $|\zeta| = 1$ assumes continuous real values $u = u(\theta)$, $(0 \leqq \theta < 2\pi)$, is given by

$$f(\zeta) = \frac{1}{\pi i} \int_\gamma \frac{u(\theta) d\sigma}{\sigma - \zeta} + \text{const.}, \tag{1}$$

where $\sigma = e^{i\theta}$, and ζ is any point interior to the unit circle γ. It is readily shown that (1) is entirely equivalent to the formula of Schwarz.

Consider a continuous, simple closed curve C in the complex z plane and let $z = w(\zeta)$ map the region bounded by C conformally on the unit circle γ in the ζ plane. Let $F(z)$ be analytic in the interior of C, and let the value of the real part of $F(z)$ on the boundary of C be $u(x, y)$.

Substituting

$$x = \frac{w(\zeta) + \bar{w}(\bar{\zeta})}{2}, \qquad y = \frac{w(\zeta) - \bar{w}(\bar{\zeta})}{2i}$$

in $u(x, y)$ gives

$$u(x, y) = \phi(\sigma, \bar{\sigma}) = \phi\left(\sigma, \frac{1}{\sigma}\right),$$

where bars denote the conjugate values. Hence, from (1),

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* Presented to the Society, December 30, 1937.

† N. Muschelišvili, Rendiconti delle Reale Accademia dei Lincei, vol. 9 (1929), pp. 295–300; Mathematische Annalen, vol. 107 (1932), pp. 282–312; Zeitschrift für Angewandte Mathematik und Mechanik, vol. 13 (1933), pp. 264–282.