SOME THEOREMS ON SUBSEQUENCES†

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It is obvious that, for any real sequence for which the sum Σ of the moduli of its elements exists and is finite, there exists a subsequence such that the modulus of the sum of its elements is not less than $\Sigma/2$. The purpose of this paper is to formulate and investigate analogous statements for complex sequences.

Let $\mathfrak{A}$ be the class of sequences, finite or infinite, $\{a_k\}$ (denoted alternatively by A) of non-zero complex numbers for which $\sum|a_k|<\infty$, and $\{a_j'\}$ (denoted alternatively by S), the general subsequence of $\{a_k\}$ for fixed $\{a_k\}$. Let $\mathfrak{B}$ be the class of sequences $\{b_k\}$ (denoted alternatively by B) of non-zero complex numbers for which $\sum|b_k|=\infty$, and $\{b_j'\}$ (denoted alternatively by T), the general subsequence of $\{b_k\}$ for fixed $\{b_k\}$.

The following facts will be established: (i) Given any sequence $\{a_k\}\epsilon\mathfrak{A}$, there then exists a subsequence $\{a_j^*\}$ for which $|\sum a_j^*| = \sup_S |\sum a_j'|$. (ii) If $\rho\equiv\inf_A \max_S |\sum a_j'|/\sum|a_k|$, then $\rho=1/\pi$. (iii) No sequence $\{a_k\}\epsilon\mathfrak{A}$ exists for which $\max_S|\sum a_j'|/\sum|a_k|=\rho$. (iv) Given any sequence $\{b_k\}\epsilon\mathfrak{B}$, there exists a subsequence $\{b_j^*\}$ such that‡

$$\limsup \Big|\sum{}' b_j^*\Big| \Big/ \sum_1^N |b_k| = \sup_T \limsup_N \Big|\sum{}' b_j'\Big| \Big/ \sum_1^N |b_k|$$

$$= \limsup_N \sup_T \Big|\sum{}' b_j'\Big| \Big/ \sum_1^N |b_k| = \limsup_N \max_T \Big|\sum{}' b_j'\Big| \Big/ \sum_1^N |b_k|.$$

(v) If $\sigma=\inf_B \max_T \limsup_N |\sum' b_j'|/\sum_1^N|b_k|$, then $\sigma=\rho$. (vi) There exists a sequence $\{b_k\}\epsilon\mathfrak{B}$ for which $\max_T \limsup_N |\sum' b_j'|/\sum_1^N|b_k|=\sigma$.

Use will be made of abbreviations of the following sort: $A_k\equiv|a_k|$, $\phi_k\equiv\arg a_k$. For definiteness, the function "arg" will mean, throughout this paper, principal argument. Given any sequence $\{a_k\}\epsilon\mathfrak{A}$, define

$$F(\phi) \equiv \sum_{\cos(\phi-\phi_k)>0} A_k \cos(\phi-\phi_k)$$
$$= \sum A_k\{\cos(\phi-\phi_k) + |\cos(\phi-\phi_k)|\}/2, \qquad 0\leqq\phi\leqq 2\pi.$$

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‡ The notation $\sum'$ indicates summation over precisely those elements of the subsequence which occur among the elements of the original sequence summed elsewhere in the formula.