ON THE ORDER OF THE PARTIAL SUMS OF A FOURIER SERIES*

W. C. RANDELS

We propose to show here that the known estimate $S_n = o(n)$ cannot be improved. To do this it is sufficient to show that there exists a sequence of functions $f_n(x)$ for which there is a constant A such that,

$$S_n(f_n, 0) > An, \tag{1}$$

and

$$S_\nu(f_n, 0) \to 0 \quad \text{as} \quad \nu \to \infty, \tag{2}$$

$$\int_{-\pi}^{\pi} | f_n(x) dx \ | < 1. \tag{3}$$

For, if (1), (2), and (3) are satisfied, then for every sequence of positive numbers d_n, with $d_n \to 0$ as $n \to \infty$, $\limsup nd_n = \infty$, we can choose a sequence of integers n_i such that

$$\left| \sum_{j=1}^{i-1} d_{n_j} S_{n_i}(f_{n_j}, 0) \right| < \frac{A}{3} n_i d_{n_i} \tag{4}$$

and

$$d_{n_{i+1}} < \frac{A}{3\pi} d_{n_i}. \tag{5}$$

We notice that

$$S_{n_j}(f_{n_i}, 0) = \frac{\pi}{2} \int_{-\pi}^{\pi} f_{n_i}(x) D_{n_j}(x) dx < \pi n_j \int_{-\pi}^{\pi} | f_{n_i}(x) | dx < \pi n_j, \tag{6}$$

and this implies that the constant A in (1) is less than π. Then, if $f(x)$ is defined by

$$f(x) = \sum_{i=1}^{\infty} d_{n_i} f_{n_i}(x),$$

$f(x) \subset L$, since from (3) and (5)

$$\int_{-\pi}^{\pi} | f(x) | dx \leqq \sum_{i=1}^{\infty} d_{n_i} \int_{-\pi}^{\pi} | f_{n_i}(x) | dx < \frac{d_1 A}{\pi} \sum_{i=1}^{\infty} 3^{-i} = \frac{d_1 A}{2\pi}.$$

* Presented to the Society, April 10, 1937.