BIRATIONAL TRANSFORMATIONS IN 4-SPACE AND 5-SPACE

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The purpose of this paper is to describe certain Cremona transformations in 4-space and 5-space, especially the former. The transformations in question are interesting on account of analogies with transformations in ordinary space, because they can be simply expressed and effectively studied by the use of equations, as well as synthetically, and because of the classic nature of the loci involved. A rational quartic curve in S_4 and a Veronese surface in S_5 are of special importance.

1. **The T_{2-4} in S_4.** Through C_1^4, a rational quartic curve in S_4, pass ∞^5 hyperquadrics. Three of these intersect in a residual curve of order 4, also rational, that meets C_1^4 six times. Therefore through the C_1^4 and a general point O of S_4 pass ∞^4 hyperquadrics, four of which have one variable point in common. Thus the ∞^4 hyperquadrics, which may be conveniently called the system (ϕ), may be put in 1:1 correspondence with the hyperplanes of the same or another S_4. The C_1^4 and the point O constitute the fundamental points of the first space, or the first fundamental system. Three ϕ's have in common a residual rational quartic c to which corresponds a line in the second space. To intersections of c with a hyperplane correspond intersections of the line with a hypersurface of the system (ϕ'). The latter are of order 4, and the transformation is described by the symbol T_{2-4}.

Let the rational quartic C_1^4 be given by $x_0:x_1:x_2:x_3:x_4=t^4:t^3:t^2:t:1$. The determinants of the matrix

$$\left\|\begin{matrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{matrix}\right\|$$

equated to zero give six linearly independent hyperquadrics containing the curve. Through a general point O $(y_0:y_1:y_2:y_3:y_4)$ of S_4 pass ∞^4 hyperquadrics that contain the curve. One of them is a cone with vertex at O. Its equation is the sum of the six products formed from the above matrix $(x_0x_2-x_1^2)(y_2y_4-y_3^2)+(x_0x_3-x_1x_2)(y_2y_3-y_1y_4)+\cdots=0$. We may, however, define the transformation thus:

$$\begin{aligned} x_0':x_1':x_2':x_3':x_4' = {} & (x_0x_2-x_1^2):(x_0x_3-x_1x_2):(x_0x_4-x_1x_3): \\ & (x_1x_4-x_2x_3):(x_2x_4-x_3^2). \end{aligned} \tag{1}$$