A QUADRATIC FORM PROBLEM IN THE CALCULUS OF VARIATIONS*

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The problem which we shall discuss arose in connection with sufficiency theorems in the multiple integral problem of the calculus of variations. It was proposed by Professor G. A. Bliss to his University of Chicago seminar (summer, 1937) and communicated to the author by Professor W. T. Reid. The result of the author's investigation presented here is a very interesting theorem on real quadratic forms.

We first have the trivial lemma:

LEMMA 1. *Let f and g be real quadratic forms in $x_1, \cdots, x_n$, and g be negative definite. Then there exists a real non-singular linear transformation carrying g and f respectively into*

$$G = -(x_1^2 + \cdots + x_n^2), \qquad F = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2,$$

where the λ_i are the roots of the determinant $|f+\lambda g| = 0$† and may be arranged so that

$$\lambda_n \geqq \lambda_{n-1} \geqq \cdots \geqq \lambda_1. \tag{1}$$

Moreover $f+\lambda g$ is positive definite if and only if

$$\lambda_1 > \lambda > -\infty. \tag{2}$$

For we may carry g into G. Apply a real orthogonal transformation carrying the resulting f into diagonal form F. The λ_i are clearly the roots of $|F+\lambda G| = 0$ and hence of $|f+\lambda g| = 0$. Finally $f+\lambda g$ is positive definite if and only if $F+\lambda G$ is positive definite, that is if $\lambda_i - \lambda \geqq \lambda_1 - \lambda > 0$.

We next derive the following lemma:

LEMMA 2. *Let g be non-singular and indefinite of index p, and let there exist a real λ_0 such that $f+\lambda_0 g = h$ is positive definite. Then there exists a real non-singular linear transformation carrying g and f respectively into*

$$\begin{aligned} G &= x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_n^2), \\ F &= -(\lambda_1 x_1^2 + \cdots + \lambda_p x_p^2) + \lambda_{p+1} x_{p+1}^2 + \cdots + \lambda_n x_n^2, \end{aligned} \tag{3}$$

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† By this we mean the determinant of the matrix corresponding to the pencil of forms $f+\lambda g$.