NOTE ON CONVEX REGIONS ON THE SPHERE*

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By a convex region on the sphere we mean a region such that any great circle arc of length less than 180°, whose end points lie in the region, lies entirely in the region. Let G denote any convex region, G_0 the diametrically opposite region, and G_1 the set obtained from the whole sphere by excluding G and G_0 together with their boundaries. Let $\rho = \arctan(3^{1/2}/2)$; then $40°53' < \rho < 40°54'$. We shall prove the following theorem.†

There is on the sphere a circle-interior of radius ρ (measured on the sphere), which lies entirely in G or entirely in G_1. The number ρ cannot be replaced by any larger number.

Let r be the least upper bound of the radii of circle-interiors lying in G. Then for every integer $n > 1$ there is a circle-interior of radius $r(1-1/n)$ lying in G. A limit point of their centers is the center of a circle-interior C of radius r lying in G. We may suppose that G is neither the whole sphere nor a hemisphere, so that $r < 90°$.

No closed semicircumference forming part of the boundary of C can be free of boundary points of G. For if it were, it would be at a distance $d > 0$ from the boundary of G. Thus G would contain not only C but all points within a distance d from one-half of its circumference. Hence G would contain a circle-interior of radius greater than r.

If P is any boundary point of G on the circumference of C, then G lies entirely on one side of the great circle tangent to C at P. For if there were a point P' of G on the opposite side of this great circle from C, we could join P' to P by a great circle arc of less than 180°, which when extended through P would cut C. Taking P'' as a point sufficiently near to P on $P'P$ extended, we should have $P'P'' < 180°$, P on $P'P''$, P' and P'' in G, therefore P in G, contrary to hypothesis.

We shall show that there is an arc PQ forming not more than one-half nor less than one-third of the circumference of C, and whose end points P and Q are boundary points of G. Let P_1 be any boundary point of G on the circumference of C, and let P_0 be the opposite point of the circumference. If P_0 is a boundary point of G, then we may take either arc P_1P_0 as PQ. Otherwise let P_2 and P_3 be the boundary

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† This theorem was suggested to me by Hans Lewy, who makes use of it in a current paper.