EXISTENCE THEOREMS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF NON-INTEGRAL ORDER*

EVERETT PITCHER AND W. E. SEWELL

1. **Introduction.** In this paper we prove theorems on the existence and uniqueness of solutions of the differential equation

$$D_x^\alpha y = \phi(x, y), \qquad \alpha > 0, \tag{1.1}$$

where $\phi(x, y)$ is a known function, $y(x)$ is an unknown function, and $D_x^\alpha y$ is the Riemann-Liouville† generalized derivative of order α of the function $y(x)$. For $\alpha=1$ the equation (1.1) is an ordinary differential equation of the first order and the restrictions on $\phi(x, y)$ for non-integral α are found to be quite similar to those imposed on the function in the integral case.

In establishing the fundamental existence theorem we first prove (§2) a theorem of the kind considered by Birkhoff and Kellogg.‡ Our proof rests on three lemmas which are contained in §3 along with the definition of the generalized derivative. In §4 we establish the existence of a unique solution in the small for $0<\alpha<1$. The extension of this solution throughout the region of definition of $\phi(x, y)$ and the case $\alpha>1$ are considered in §§5 and 6 respectively.

2. **The general existence theorem.** For our purposes the following theorem is fundamental:

THEOREM 2.1. *Let E be a set of continuous functions defined on a common closed interval, and such that if a sequence of functions each belonging to E is uniformly convergent, then the limiting function belongs to E also. Let S be an operator such that if y is in E, then Sy is in E,*

* Presented to the Society, September 10, 1937.

† B. Riemann, *Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass*, Leipzig, 1892, pp. 331–344; J. Liouville, *Sur quelques questions de géométrie et de mécanique, et un nouveau genre de calcul pour résoudre ces questions*, Journal de l'École Polytechnique, (1), vol. 13, no. 21 (1832), pp. 1–69. For further references see W. E. Sewell, *Generalized derivatives and approximation by polynomials*, Transactions of this Society, vol. 41 (1937), pp. 84–123; we refer to this paper as SI. See also W. Fabian, *Expansions by the fractional calculus*, Quarterly Journal of Mathematics, Oxford Series, vol. 7 (1936), pp. 252–255, where other references are given.

‡ G. D. Birkhoff and O. D. Kellogg, *Invariant points in function space*, Transactions of this Society, vol. 23 (1922), pp. 96–115.