AN APPLICATION OF SCHLÄFLI'S MODULAR EQUATION TO A CONJECTURE OF RAMANUJAN†

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In 1918 Ramanujan‡ made the following conjecture:

If $q=5$, 7, or 11, and if $24n-1$ is divisible by q^α, then the number $p(n)$ of unrestricted partitions of n is divisible by q^α.

Ramanujan himself proved this conjecture to be true in case‡ $q^\alpha=5$, 7, 5^2, and 7^2, and also§ for $q^\alpha=11$ and 11^2. It has since been proved‖ for $q^\alpha=5^3$. Some modification of the conjecture is necessary, however, since, as Chowla¶ was first to notice, it fails for $q^\alpha=7^3$. In fact, since $24\cdot 243-1=5831$ is divisible by 7^3, it would follow from the conjecture that $p(243)$ is also divisible by 7^3. However, Gupta's table** of $p(n)$ gives

$$p(243) = 13397\ 82593\ 44888,$$

a number†† which is not divisible by 7^3. It occurred to the writer that it would be worth while making an investigation of $p(599)$ and $p(721)$ relative to their divisibility by 5^4 and 11^3 respectively.‡‡ To obtain the value of $p(n)$ for these isolated values of n beyond the limits of then existing tables, use was made of the celebrated Hardy-Ramanujan series,§§ which may be written

$$(1)\qquad p(n) = \frac{(12)^{1/2}}{\mu(24n-1)}\sum_{k=1}^{N} A_k^*(n)(\mu-k)e^{\mu/k} + r_n(N),$$

where we have written μ for $\pi(24n-1)^{1/2}/6$. By taking $N=18$ for $n=599$, and $N=21$ for $n=721$, values were obtained for the series in

† Presented to the Society, September 10, 1937.

‡ Proceedings of the London Mathematical Society, vol. 19 (1919), pp. 207–210; *Collected Papers*, pp. 210–213.

§ Mathematische Zeitschrift, vol. 9 (1921), pp. 147–153; *Collected Papers*, pp. 232–238. A proof for 11^2 is in one of his notebooks.

‖ See Bulletin of the Academy of Sciences, U.R.S.S., 1933, no. 6, pp. 763–800.

¶ Journal of the London Mathematical Society, vol. 9 (1934), p. 247.

** Proceedings of the London Mathematical Society, (2), vol. 39 (1935), p. 149.

†† This number has been verified by the present writer.

‡‡ Journal of the London Mathematical Society, vol. 11 (1936), pp. 114–118.

§§ Proceedings of the London Mathematical Society, (2), vol. 17 (1918), pp. 75–115. Ramanujan's *Collected Papers*, pp. 276–309.