A NOTE ON LINEAR TOPOLOGICAL SPACES*

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A space T is called a linear topological space if (1) T forms a linear† space under operations $x+y$ and αx, where $x,y \epsilon T$ and α is a real number, (2) T is a Hausdorff topological space,‡ (3) the fundamental operations $x+y$ and αx are continuous with respect to the Hausdorff topology. The study§ of such spaces was begun by A. Kolmogoroff (cf. [4]. Kolmogoroff's definition of a linear topological space is equivalent to that just given). Kolmogoroff calls a set $S \subset T$ bounded, if for any sequence $x_\nu \epsilon S$ and any real sequence α_ν converging to 0 we have $\lim_{\nu\to\infty} \alpha_\nu x_\nu = \theta$, where θ is the zero element of T. He then shows that a linear topological space T reduces to a linear normed space‖ if and only if there exists in T an open set which is both convex¶ and bounded. In this note, the characterization of other types of spaces among the class of linear topological spaces is studied. Spaces which are *locally bounded*, that is spaces containing a bounded open set, are found to be "pseudo-normed" on the one hand, and metrizable on the other, but not in general normed. Fréchet spaces, or spaces of type (F), are characterized. The main result of the paper is that a linear topological space T is finite dimensional, and hence linearly homeomorphic to a finite dimensional euclidean space, if and only if T contains a compact, open set. This of course is a generalization to linear topological spaces of the well known theorem of F. Riesz for the space of continuous functions.

We first give some needed properties of bounded sets. The following notations will be used throughout. We denote by αS the set of all αx with $x \epsilon S$; by $x+S$, the set of all elements $x+y$ where y ranges over S; by S_1+S_2, the set of all $x+y$ with $x \epsilon S_1$, $y \epsilon S_2$.

THEOREM 1. *A set S of a linear topological space is bounded if and only if, given any neighborhood U of the origin, there is an integer ν such that $|\alpha| < 1/\nu$ implies $\alpha S \subset U$.*

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† Cf., for example, [1], p. 26. Numbers in brackets refer to the bibliography at the end of the paper.

‡ Cf. [2], pp. 228–229, axioms (A), (B), (C), (5); or [3], pp. 43 and 67.

§ Important instances of a linear topological space were studied by J. von Neumann several years before Kolmogoroff's paper was published. Cf. Mathematische Annalen, vol. 102 (1930), pp. 370–427. See also [5].

‖ Cf. [1], p. 53.

¶ A set S is convex if and only if $x,y \epsilon S$ and $0 < \alpha < 1$ imply $\alpha x + (1-\alpha) y \epsilon S$.