TOPOLOGICAL PROPERTIES OF DIFFERENTIABLE MANIFOLDS*

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I. INTRODUCTION

1. *The Problems* In many fields of work one is led to the consideration of n-dimensional spaces. A given dynamical system has a certain number of "degrees of freedom"; thus a rigid body, with one point fixed, has three. A line in euclidean space is determined by four "parameters." We therefore consider the positions of the rigid body, or the straight lines, as forming a space of three, or four, dimensions.† But when we try to determine the points of the space by assigning to each a set of three, or four, numbers, we are doomed to failure. This is possible for a small region of either space, but not for the whole space at once. The best we can do is to cover the space with such regions, define a coordinate system in each, and state how the coordinate systems are related in any two overlapping regions. They will be related in general by means of differentiable,‡ or analytic, transformations, with non-vanishing Jacobian. Any such space we shall call a differentiable, or analytic, manifold.

For a complete study of such spaces, we must know not only properties of euclidean n-space E^n, which we may apply in each coordinate system separately, but also properties which arise from the manifold being pieced together from a number of such systems. It is these latter properties, essentially topological in character, which form the subject of the present address.

Suppose we wish to study differential geometry in the n-dimensional manifold M^n. At each point p of M^n, the possible differentials (or "tangent" vectors) form an n-dimensional vec-

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† The first space forms the group G^3 of rotations in 3-space; it is homeomorphic with projective 3-space P^3. The second space is homeomorphic with the total space (see §2) of the tangent vector space of the projective plane P^2, or of the 2-sphere S^2 if we use oriented lines. This is easily seen by considering together all parallel lines.

‡ Differentiable will always mean continuously differentiable.