NOTE ON A THEOREM CHARACTERIZING GEODESIC ARCS IN COMPLETE, CONVEX METRIC SPACES

BY L. M. BLUMENTHAL

1. *Introduction.* In his four *Untersuchungen über allgemeine Metrik*,* Menger initiated the systematic study of the metric geometry of abstract semi-metric and metric spaces. Among the most important of the notions Menger introduced in such spaces is that of *convexity* which leads, in complete metric spaces, to the existence of geodesic arcs joining each pair of points a, b of the space. Such an arc is congruent to a line segment of length ab.†

Concerning geodesic arcs in complete, convex metric spaces, Menger gives the following theorem:‡

THEOREM. *The geodesic arcs joining two points a, b of a complete, convex metric space are characterized among all arcs joining a, b by the following property: if p, q are elements of a geodesic arc joining a, b (p, q both distinct from a, b) then either p is between§ a and q, or p is between q and b, or p is identical with q.*

That a geodesic arc joining a, b has this property follows directly, as Menger observes, from the fact that such an arc may be imbedded congruently in a line segment of length ab. To show, however, that the property is *characteristic* for geodesic arcs it must be shown, of course, that every arc joining a, b that has this property is a geodesic arc. This sufficiency of the property is not shown by Menger (two proofs of the *necessity* of the condition being given instead). As the theorem is of use in developing some of the properties of convex spaces, and as a search of the literature, as well as conversation with Menger, has re-

* Mathematische Annalen, vol. 100 (1928), pp. 75–163; vol. 103 (1930), pp. 466–501.

† The term "geodesic arc" is used by the author in the geometry of distances always in the sense of minimizing geodesic.

‡ *Erste Untersuchung*, loc. cit., p. 91.

§ A point q lies between two points p, r if and only if $p\neq q\neq r$ and $pq+qr=pr$. We symbolize this relation by writing pqr.