INEQUALITIES SATISFIED BY A CERTAIN DEFINITE INTEGRAL

BY G. H. HARDY AND NORMAN LEVINSON*

1. *Introduction.* In this note we solve the following problem. Suppose that

$$0 \leqq a_1 < a_2 < \cdots < a_{2n+1} \leqq 1,$$

$$(1) \qquad f(x) = \frac{(x-a_2)(x-a_4)\cdots(x-a_{2n})}{(x-a_1)(x-a_3)\cdots(x-a_{2n+1})},$$

$$J(t) = \int_0^1 |f(x)|^t dx, \qquad 0 < t < 1.$$

Then what are the best inequalities satisfied by $J(t)$?

We prove the following theorem:

THEOREM A. *If $f(x)$ satisfies* (1) *then*

$$\frac{\Gamma(\frac{1}{2}+\frac{1}{2}t)\Gamma(1-\frac{1}{2}t)}{(1-t)\pi^{1/2}} \leqq J(t) \leqq \frac{2^t}{1-t},$$

with inequality except when

$$f(x) = \frac{1}{x-\frac{1}{2}}, \qquad J(t) = \frac{2^t}{1-t};$$

$$f(x) = \frac{x-\frac{1}{2}}{x(x-1)}, \qquad J(t) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}t)\Gamma(1-\frac{1}{2}t)}{(1-t)\pi^{1/2}}.$$

The integral $J(t)$ occurred in a recent paper by Levinson.† Levinson proved that

$$J(t) < \frac{5}{1-t},$$

and indeed that

$$\int_0^1 |f(x+iy)|^t < \frac{5}{1-t}$$

* National Research Fellow.

† Levinson, *On non-harmonic Fourier series*, Annals of Mathematics, (2), vol. 37 (1936), p. 922.