A NOTE ON THE CESÀRO METHOD OF SUMMATION*

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1. *Introduction.* A sequence $\{S_n\}$, or a series $\sum U_n$ with partial sums S_n, is said to be summable by the Cesàro mean of order α, or summable (C, α), to the sum s, if $\sigma_n^\alpha = S_n^\alpha / A_n^\alpha \to s$,† where S_n^α and A_n^α are given by the following relations:

$$(1) \quad (1 - x)^{-\alpha-1} = \sum A_n^\alpha x^n; \quad A_n^\alpha = \frac{(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)}{n!};$$

$$(2) \qquad \sum S_n^\alpha x^n = (1 - x)^{-\alpha} \sum S_n x^n = (1 - x)^{-\alpha-1} \sum U_n x^n;$$

$$S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} S_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} U_\nu;$$

and where α is any complex number other than a negative integer.‡ We shall restrict ourselves in this note to real orders of summability. It is known that if a sequence or series S is summable (C, α), $\alpha > -1$, it is summable (C, α'), $\alpha' > \alpha$, to the same sum.§ If a sequence or series S is summable (C, α) for all $\alpha \geqq \gamma$, then the lower limit of all such possible values of γ is called by Chapman‖ the *index of summability* of S.

It is sometimes easier to find the indices of summability and the sums of certain subsequences of a sequence S than to find the index and sum of S itself. As a trivial example, let $\{S_n\}$ be the sequence of partial sums of Leibniz's series $1-1+1-1+ \cdots$. Then $S_{2k}=1$, $S_{2k+1}=0$, and it is easily seen that $\{S_{2k}\}$ is summable to the value 1 and $\{S_{2k+1}\}$ to the value 0 by the Cesàro mean of any order. It is the purpose of this note

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† Superscripts will not denote exponents when applied to capital letters and to the letter σ.

‡ For a systematic account of the Cesàro method, see Kogbetliantz, *Summation des Séries et Intégrales Divergentes par les Moyennes Arithmétiques et Typiques*, Paris, 1931.

§ Kogbetliantz, op. cit., p. 17.

‖ Proceedings of the London Mathematical Society, (2), vol. 9 (1911), pp. 369–409; p. 378.