# THE MOST GENERAL TRANSFORMATIONS OF PLANE REGIONS WHICH TRANSFORM CIRCLES INTO CIRCLES

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1. *Introduction.* If we consider two plane regions $R$ and $R'$ which are mapped conformally the one upon the other, there corresponds to every circle $c$ contained in $R$ an analytic closed curve $c'$ contained in $R'$. If an arc $\alpha$ lying on the circle $c$ has a circular image the curve $c'$ must be itself a circle.

Suppose that the interior of the circle $c$ belongs to the region $R$. Then both circular discs bounded by $c$ and $c'$ respectively are represented conformally the one upon the other. It is a well known fact that in this case the transformation of these circles into one another is given by a transformation of M bius by which, furthermore, every circle of $R$ is transformed into a circle of $R'$.

If we drop now the condition that the one to one correspondence of our regions must be conformal, the assumption that one single circle of $R$ has a circular image is no longer sufficient to characterize the transformations of Möbius. On the other hand, if we make the stronger assumption that *every* closed circle contained in $R$ is transformed into a circle, we shall see in the course of this paper that a theorem analogous to the one stated above holds under surprisingly general conditions. To prove that the transformation which we consider is a transformation of Möbius it is no longer necessary to assume from the outset (as is the case for the analogous theorem concerning collinear transformations) that this transformation is continuous, or that it is measurable in the sense of Lebesgue, or even that the point set $R'$ is itself a region.

*The condition that circles lying in a region R have circular images characterizes the group of Möbius transformations among all the one to one arbitrary correspondences between the points of R and the points of a quite arbitrary point set of the plane.*

2. *Statement of a Preliminary Theorem.* We consider a circular open disc which we shall call $C$ and suppose that there is a one