BILINEAR FUNCTIONALS ON THE SPACE OF BOUNDED, MEASURABLE FUNCTIONS

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1. *Introduction.* We propose to make an extension of the Radon integral to two variables, and by properly defining the notion of limited variation to show that every continuous bilinear functional on the class of bounded, measurable functions (with suitably defined norm) is expressible in terms of this integral. Conversely we show that the integral is bilinear and continuous. The paper is concluded with a theorem establishing a relation between the double and iterated integrals. Similar results for the case of continuous linear functionals have recently been obtained.*

2. *Preliminary Definitions.* Let $\mathfrak{S}$ be the class of all measurable subsets of the interval $\mathfrak{J}=[0\leqq x\leqq 1]$, and let ϕ be a function on $\mathfrak{S}\mathfrak{S}$ to $\mathfrak{A}$, the set of all real numbers. Such a function will be termed *bi-additive* in case, for every two disjoint sets σ_1, σ_2, we have $\phi(\sigma, \sigma_1+\sigma_2)=\phi(\sigma, \sigma_1)+\phi(\sigma, \sigma_2)$ and $\phi(\sigma_1+\sigma_2, \sigma)=\phi(\sigma_1, \sigma)+\phi(\sigma_2, \sigma)$; the function ϕ will be said to be of *limited variation* in case the upper bound of

$$(1)\qquad \left| \sum_{i,j} \epsilon_{1i}\phi(\sigma_{1i}, \sigma_{2j})\epsilon_{2j} \right|$$

for all divisions $\sigma_{k_1}, \cdots, \sigma_{km_k}$, $(k=1, 2)$, of $\mathfrak{J}$ into disjoint sets and all numbers ϵ_{ki} such that $|\epsilon_{ki}|=1$, $(k=1, 2; i=1, 2, \cdots, m_k)$, is finite;† we shall designate this bound, when it is finite, by $T\phi$.

Let μ_1, μ_2 be any two functions in $\mathfrak{M}$, the class of bounded, measurable functions, such that $d_{k0}\leqq\mu_k(x)<D_k$, $(k=1, 2; 0\leqq x\leqq 1)$. Then following the usual procedure in the Lebesgue

* See, T. H. Hildebrandt, *On bounded linear functional operations*, Transactions of this Society, vol. 36 (1934), p. 868; or G. Fichtenholz and L. Kantorovitch, *Sur les opérations dans l'espace des fonctions bornées*, Studia Mathematica, vol. 5 (1934), p. 69.

† This definition is a generalization to set functions of Fréchet's definition for functions of real variables; see, M. Fréchet, *Sur les fonctionelles bilinéaires*, Transactions of this Society, vol. 16 (1915), p. 215.