# A NOTE ON YOUNG-STIELTJES INTEGRALS*

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THEOREM 1. *If $f(x)$ is bounded and measurable Borel, and $g_1(x)$, $g_2(x)$ are of bounded variation, then the following equality holds*:

$$\int_0^1 f(x)d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x+0)dg_2(x) + \int_0^1 f(x)g_2(x-0)dg_1(x). \tag{1}$$

PROOF. In a recent article Evans† showed that if $g_1(x)$ and $g_2(x)$ have no common points of discontinuity, then

$$\int_0^1 f(x)d[g_1(x)g_2(x)] = \int_0^1 f(x)g_1(x)dg_2(x) + \int_0^1 f(x)g_2(x)dg_1(x).$$

Therefore (1) holds if either $g_1(x)$ or $g_2(x)$ are continuous. It remains to show that the theorem holds when $g_1(x)$ and $g_2(x)$ are both step functions. Under these circumstances we have

$$\begin{aligned}\int_0^1 f(x)g_1(x+0)dg_2(x) &+ \int_0^1 f(x)g_2(x-0)dg_1(x)\\ &= \sum f(\alpha_i)g_1(\alpha_i+0)[g_2(\alpha_i+0) - g_2(\alpha_i-0)]\\ &\quad + \sum f(\alpha_i)g_2(\alpha_i-0)[g_1(\alpha_i+0) - g_1(\alpha_i-0)]\\ &= \sum f(\alpha_i)[g_1(\alpha_i+0)g_2(\alpha_i+0) - g_1(\alpha_i-0)g_2(\alpha_i-0)]\\ &= \int_0^1 f(x)d[g_1(x)g_2(x)],\end{aligned}$$

where the summations are taken over all the discontinuities of $g_1(x)$ and $g_2(x)$.

The following lemmas are immediate applications of equation (1).

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† G. C. Evans, *Correction and addition to "Complements of potential theory,"* American Journal of Mathematics, vol. 57 (1935), pp. 623–626.