A SIMPLIFIED SOLUTION OF THE EQUATION $\Delta y(x) = F(x)$*

BY I. M. SHEFFER

We have elsewhere considered, from the point of view of a local solution, the equation

$$(1) \qquad \Delta y(x) \equiv y(x+1) - y(x) = F(x),$$

and certain related equations.† It has now been found possible to simplify the form of the solution of (1) by altering the method. This we propose to show in the present note. Suitably modified, this method may well be expected to apply to other equations.

Let Δ be the difference operator. If t is a parameter, then

$$(2) \qquad \Delta[e^{tx}] = e^{tx}(e^t - 1),$$

so that

$$(3) \qquad e^{tx}(e^t - 1) = \sum_{n=0}^{\infty} P_n(x)t^n,$$

where $P_n(x)$ is the polynomial defined by

$$(4) \qquad P_n(x) = \Delta\left[\frac{x^n}{n!}\right] = \frac{1}{n!}\left\{(x+1)^n - x^n\right\}.$$

Suppose that the coefficient of t^n is multiplied (for every n) by $n!$ on both sides of (3). This yields

$$(5) \qquad \frac{1}{1 - t(x+1)} - \frac{1}{1 - tx} = \sum_0^{\infty} n!P_n(x)t^n.$$

We now transform (5) by replacing t by $1/t$ and dividing through by t:

$$(6) \qquad \frac{1}{t - 1 - x} - \frac{1}{t - x} = \sum_0^{\infty} \frac{n!P_n(x)}{t^{n+1}}.$$

---

* Presented to the Society, September 5, 1936.

† *A local solution of the difference equation $\Delta y(x) = F(x)$ and of related equations*, Transactions of this Society, vol. 39 (1936), pp. 345–379.