ON LINE INTEGRALS AND DIFFERENTIAL EQUATIONS, ESPECIALLY THOSE OF DYNAMICS

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1. *Introduction and Formulation of the Problem.* Recently a formula was given, exhibiting a Cartan relatively invariant line integral for a non-holonomic conservative dynamical system.* I wish to show that this formula is a special case of a more general formula which will here be developed. The formula itself is not explicitly that of a relatively invariant line integral, even in the special case previously treated. Nevertheless, under certain restrictions it can undoubtedly be put into that form.†

Let us consider the system of differential equations

$$(1)\qquad \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_m}{X_m},$$

together with an arbitrary linear differential form $\sum_{i=1}^{m} A_i dx_i$, where the $X_i(x_1, \cdots, x_m)$ and $A_i(x_1, \cdots, x_m)$ are of class C'' in a region R. Let Γ_0 be a closed curve‡ in R, whose parametric equations are given, say, by

$$x_i = x_i^{(0)}(\tau), \quad 0 \leqq \tau \leqq 1, \quad x_i^{(0)}(0) = x_i^{(0)}(1), \quad (i = 1, \cdots, m).$$

Consider the *tube* T of trajectories of (1) which pass through Γ_0. Let Γ_{u_1} be any other similar closed curve embracing this tube; that is, Γ_{u_1} cuts each trajectory of the tube the same number of times as Γ_0 and in the same order.§ The whole of T between Γ_0 and Γ_{u_1} inclusive is assumed to be contained in R.

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* A. E. Taylor, *On the integral invariants of non-holonomic dynamical systems*, this Bulletin, vol. 40 (1934), pp. 735–742.

† For a complete treatment and bibliography of the related subject of integral invariants see E. J. Cartan's book, *Leçons sur les Invariants Intégraux*, 1922.

‡ The curve Γ_0 may have double points. For definiteness it may be assumed that the $x_i^{(0)}(\tau)$ are of class C'' except for a finite number of corners.

§ It is to be noticed that T is a *singular tube* (in a certain obvious sense) whenever Γ_0 cuts a trajectory more than once. This is necessarily the case when $m=2$.