AN ARITHMETIC FUNCTION

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1. *Introduction*. The function*

$$\psi(k, m) = \sum_{st=k} \mu(s) m^t, \tag{1}$$

where $\mu(s)$ is the Möbius function, has the property

$$\psi(k, m) \equiv 0 \pmod{k}, \tag{2}$$

for arbitrary integral m. Gegenbauer† has generalized this by replacing $\mu(s)$ by an arbitrary integral-valued function $w(s)$ for which

$$\sum_{s \mid k} w(s) \equiv 0 \pmod{k}, \tag{3}$$

for all k. Clearly (3) holds for the function $\mu(s)$. Since (1) is equivalent to

$$\sum_{s \mid k} \psi(s, m) = m^k,$$

we put

$$W(k, m) = \sum_{st=k} w(s) m^t = \sum_{sde=k} w(s)\psi(e, m) = \sum_{te=k} \psi(e, m) \sum_{s \mid t} w(s) \tag{4}$$

and therefore by (2) and (3),

$$W(k, m) \equiv 0 \pmod{k}, \tag{5}$$

for all m. Conversely it is easy to show, by an induction on k, that (5) implies (3). Indeed, if (3) holds for all integers $<k$, it follows from (4) and (5) that

$$\psi(1, m) \sum_{s \mid k} w(s) = m \sum_{s \mid k} w(s) \equiv 0 \pmod{k}.$$

Since this must hold for all m, we may select an m prime to k, and therefore we have (3).

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* For references see Dickson's *History of the Theory of Numbers*, vol. 1, pp. 84–86. Cited as Dickson.

† See Dickson, p. 86.