# A GENERALIZATION OF SCHWARZ'S LEMMA*

BY CONSTANTIN CARATHÉODORY

1. *Introduction.* We consider the family of functions $f(z)$, which are regular inside of the unit circle, which vanish at the origin, and whose absolute value $|f(z)|$ is less than one in that circle. Taking two points $z_1$ and $z_2$ in the interior of the unit circle we inquire about the maximum $M(z_1, z_2)$ of the expression

$$\left| \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right| \tag{1}$$

if $f(z)$ describes the family of functions considered above.

This maximum can never be less than one, because the function $f(z) \equiv z$ itself is contained among the functions of our family. But in a great number of cases $M(z_1, z_2)$ is *exactly equal to one*. Thus if $z_1$ is taken equal to zero, the assertion that $M(0, z_2) = 1$ is only another way of formulating the lemma of Schwarz. Again, if we assume that the ratio $z_2/z_1$ is real and negative, we have

$$| f(z_2) - f(z_1) | \leqq | f(z_1) | + | f(z_2) |,$$
$$| z_2 - z_1 | = | z_1 | + | z_2 |;$$

and, using the lemma of Schwarz, we find that $M(z_1, z_2) = 1$.

In the third place, we have $M(z_1, z_2) = 1$ if *both* points $z_1$ and $z_2$ lie on the circular disc $|z| \leqq 2^{1/2} - 1$. This is an easy consequence of the fact that for all points of this figure the expression $|f'(z)|$ is never greater than one.† We are going to analyze the questions which arise from these different examples by determining completely all the cases for which $M(z_1, z_2) = 1$.

2. *An Auxiliary Function.* We begin with the obvious remark that our result will not be altered if we neglect from the outset all the functions of the form $f(z) = e^{i\theta} z$ for which the ex-

---

* From an address delivered before the Society under the title *Bounded analytic functions*, on November 27, 1936, by invitation of the Program Committee.

† J. Dieudonné, *Recherches sur quelques problèmes relatifs aux polynomes et aux fonctions bornées d'une variable complexe*, Annales de l'École Normale, (3), vol. 48 (1931), pp. 247–358; in particular, p. 352.