DISTANCE FUNCTIONS AND THE METRIZATION PROBLEM*

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1. *Introduction.* The metrization problem† is concerned with conditions under which a topological space is metrizable, that is, is homeomorphic to a metric space. A space is metric if to every two points a and b, a non-negative real number ab is assigned satisfying the well known conditions:

I. $ab=0$ *if and only if* $a=b$;

II. $ab=ba$, (symmetry);

III. $ac\leqq ab+bc$, (triangle property).

A metrization theorem is usually proved by actually introducing such a distance function into the space. However, it is often easier to introduce first into a topological space a distance function satisfying the following conditions IV or V instead of III:

IV. *If* $ab<\epsilon$ *and* $cb<\epsilon$, *then* $ac<2\epsilon$ (generalized triangle property);

V. *For every* $\epsilon>0$ *there exists* $\phi(\epsilon)>0$ *such that if* $ab<\phi(\epsilon)$ *and* $cb<\phi(\epsilon)$, *then* $ac<\epsilon$ (uniformly regular).

Condition V reduces to IV if $\phi(\epsilon)=\epsilon/2$. Chittenden‡ has shown that a space with a distance function satisfying I, II, and V is metrizable. Chittenden's proof is somewhat long and complicated. Furthermore, while the existence of a distance function satisfying III is proved, it is not defined directly in terms of the original distance function satisfying V. Alexandroff and Urysohn§ make use of Chittenden's theorem introducing a metric satisfying IV. Niemytski‖ and W. A. Wilson¶ make use of Alexandroff and Urysohn's result.

Without relying on Chittenden's theorem, the present paper gives a simple, direct proof that a topological space with a distance function satisfying I, II, IV is metrizable. The method

* Presented to the Society, September 1, 1936.

† See Chittenden, this Bulletin, vol. 33 (1927), pp. 13–34.

‡ Transactions of this Society, vol. 18 (1917), p. 161.

§ Comptes Rendus, vol. 177 (1923), p. 1274.

‖ Transactions of this Society, vol. 29 (1927), p. 507.

¶ American Journal of Mathematics, vol. 53 (1931), p. 361.