ON THE EXPANSION OF A FUNCTION ANALYTIC AT DISTINCT POINTS

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1. *Introduction.* Let $a_1, a_2, \cdots, a_\nu$ be a set of distinct points in the complex plane.* If $\theta_k(x)$ is any function which is analytic at a_k and has a simple zero at a_k, then for any sufficiently small positive number δ the level curve $|\theta_k(x)| = \delta$ will have one branch, $C_\delta^{(k)}$, which satisfies the conditions (a) a_k is interior to the curve $C_\delta^{(k)}$, (b) $C_\delta^{(k)}$ is a simple closed analytic curve, (c) a_k is the only zero of $\theta_k(x)$ either inside or on $C_\delta^{(k)}$. Furthermore, if δ' and δ'' are any two values of δ such that $\delta'' > \delta'$, then $C_{\delta''}^{(k)}$ contains $C_{\delta'}^{(k)}$ in its interior.

Let $F_{n,m}(x)$, $(n = 0, 1, 2, \cdots; m = 1, 2, \cdots, \nu)$, be a set of functions all of which are holomorphic in and on each of the curves $C_R^{(k)}$, and let each function have a zero of order n at a_m and a zero of higher order than n at all the other points a_k, $(k \neq m)$. Then in and on $C_R^{(m)}$, we have the Bürmann series†

$$(1) \qquad F_{n,m}(x) = \sum_{s=0}^{\infty} c_{n,s}^{(m,m)} \left[\theta_m(x)\right]^{n+s} \equiv \left[\theta_m(x)\right]^n P_{n,m,m}(x),$$

where $c_{n,0}^{(m,m)} \neq 0$. In and on $C_R^{(k)}$, $(k \neq m)$, we have

$$(2) \qquad F_{n,m}(x) = \sum_{s=0}^{\infty} c_{n,s}^{(m,k)} \left[\theta_k(x)\right]^{n+s+1} \equiv \left[\theta_k(x)\right]^{n+1} P_{n,m,k}(x).$$

We assume that these functions have been normalized so that $c_{n,0}^{(m,m)} = 1$. Denote by $b_{n,s}$, $(n = 0, 1, \cdots; s = 0, 1, \cdots)$, any set of positive numbers, independent of k, such that $b_{n,0} = 1$ and

$$b_{n,s} \geqq \left| \sum_{m=1}^{\nu} c_{n,s-1+\delta(m,k)}^{(m,k)} \right|$$

for $s > 0$, where $\delta(m, k)$ is Kronecker's symbol. Put

* In the work to follow the points a_k are assumed finite but the extension to the case where one point a_k is the point at infinity is immediate.

† Whittaker and Watson, *Modern Analysis*, 4th ed., p. 131.