ON THE SUMMABILITY OF FOURIER SERIES

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1. *Introduction.* It is well known that the Abel method of summability is stronger than the Cesàro methods of any order. An example has been given* to show that there are series which are Abel summable but not Cesàro summable for any order. This series is one for which $a_n \neq o(n^\alpha)$ for any α, and hence which cannot be (C, α) summable for any α. This series cannot be a Fourier series since for all Fourier series $a_n = o(1)$. We propose to give an example of the existence of a Fourier series which is Abel summable but not Cesàro summable.

We shall make use of some results of Paley† which show that, if the Fourier series of $f(x)$,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

is (C, α) summable at the point x, then, for $\beta > \alpha$,

$$\begin{aligned} R_\beta(f, t) &= \beta \int_0^t \{f(x+\tau) + f(x-\tau) - 2f(x)\}(t-\tau)^{\beta-1} d\tau \\ &= o(t^\beta), \qquad \text{as} \qquad t \to 0, \end{aligned}$$

and conversely, if $R_\alpha(f, t) = o(t^\alpha)$, as $t \to 0$, then the series (1) is (C, β) summable for every $\beta > \alpha + 1$. We shall first show that for every $n > 1$ there is a function $f_n(x)$ such that at $x = 0$

$$\overline{\lim_{t\to 0}} \left| \frac{1}{t^j} R_j(f_n, t) \right| = \infty, \qquad (j \leqq n-1), \tag{2}$$

but

$$R_n(f_n, t) = o(t^n), \qquad \text{as} \qquad t \to 0. \tag{3}$$

This implies that the Fourier series of $f_n(x)$ is $(C, n+2)$ summable at $x = 0$ and therefore Abel summable. The function

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* See Landau, *Darstellung und Begründung einiger neuer Ergebnisse der Funktionentheorie*, 1929, p. 51.

† R. E. A. C. Paley, *On the Cesàro summability of Fourier series and allied series*, Proceedings of the Cambridge Philosophical Society, vol. 26 (1929), pp. 173–203.