ON HIGHER DERIVATIVES OF ORTHOGONAL POLYNOMIALS*

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1. *Introduction.* Let $\{\phi_n(x)\}$ be a set of orthogonal polynomials in a finite interval (a, b) with the integrable (L) weight function† $p(x)$, that is,

$$\int_a^b p(x)\phi_n(x)\phi_m(x)dx = 0, \qquad (n \neq m),$$

$$p(x) \geqq 0, \quad \int_a^b p(x)dx > 0, \quad \phi_n(x) = x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n0}.$$

It has been shown‡ that if the first derivatives $\{\phi_n'(x)\}$ also form a set of orthogonal polynomials, then the original set are Jacobi polynomials. The purpose here is to show that if the rth derivatives $\{\phi_n{}^r(x)\}$ form an orthogonal set, then again $\{\phi_n(x)\}$ is a set of Jacobi polynomials. The proof is based on the following lemma.§

LEMMA. *Let $Q(x)$ be non-negative in the (finite or infinite) interval (c, d), and such that the constants β defined by the formula*

$$\beta_k = \int_c^d Q(x)x^k dx, \qquad (k = 0, 1, \cdots),$$

exist, and for a certain positive integer r

$$\int_c^d Q(x)\phi_n(x)G_{n-r-1}(x)dx = 0, \qquad (n = r + 1, r + 2, \cdots),$$

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† There is no restriction in assuming (as we do) that if α, β are any two numbers, $a<\alpha<b<\beta$, then $p(x)\neq 0$ almost everywhere in (a, α); $p(x)\neq 0$, almost everywhere in (β, b).

‡ W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638. H. L. Krall, *On derivatives of orthogonal polynomials*, this Bulletin, vol. 42 (1936), pp. 423–428.

§ See Krall, loc. cit.