NOTE ON AN ASSOCIATIVE DISTRIBUTIVE ALGEBRA IN WHICH THE COMMUTATIVE LAW OF ADDITION DOES NOT HOLD

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1. *Introduction.* We shall give here a description of an algebra in which the elements are closed under the associative laws of addition and multiplication, a left and a right distributive law holds connecting addition and multiplication, and the usual equality axioms hold, yet the commutative law of addition does not hold.

2. *Certain Matrices.* As in other papers,* we define a semi-group as a set of elements closed under an associative operation, and a semi-ring as an algebra of the type above described. In the papers just cited we considered a set of elements

$$C_1, C_2, \cdots, C_i, \cdots, C_j; \qquad (j \geqq i), \tag{1}$$

with $C_{j+1} = C_i$ and forming a semi-ring, with

$$C_m + C_n = C_{m+n}, \qquad C_m \cdot C_n = C_{mn}.$$

On page 581 of the first paper mentioned, we showed how to adjoin a zero element to this set, say C_0.

In (1), set $i=j=1$, and consider the only two distinct elements in the set, after adjoining C_0. We then have the relations

$$\begin{gathered} C_1 + C_1 = C_1; \quad C_1 + C_0 = C_0 + C_1 = C_1; \quad C_0 + C_0 = C_0; \\ C_1 \cdot C_1 = C_1; \quad C_0 \cdot C_1 = C_0 \cdot C_1 = C_0. \end{gathered} \tag{2}$$

Note that these elements are *not* isomorphic to the two residue classes modulo 2, and the cancellation law of addition does not hold in general; that is, from $C_1 + C_0 = C_1 + C_1$, we cannot infer $C_0 = C_1$. Consider the matrices

$$A = \begin{pmatrix} C_0 & C_1 \\ C_0 & C_1 \end{pmatrix}; \qquad B = \begin{pmatrix} C_1 & C_0 \\ C_1 & C_0 \end{pmatrix}; \qquad C = \begin{pmatrix} C_1 & C_1 \\ C_1 & C_1 \end{pmatrix}.$$

* Proceedings of the National Academy of Sciences, vol. 20 (1934), p. 581; vol. 21 (1935), p. 162. This Bulletin, vol. 40 (1934), pp. 916–920.