NOTE ON DIVISIBILITY SEQUENCES

BY MORGAN WARD

1. *Introduction.* We call a sequence of rational integers

$$(u): \qquad u_1, u_2, u_3, \cdots, u_n, \cdots$$

a *divisibility sequence* if u_r divides u_s whenever r divides s. The divisibility sequences most frequently studied are the *linear* sequences which satisfy linear difference equations with constant, integral coefficients.* In particular, the divisibility sequence associated with a difference equation of order two is essentially one of the important functions of Lucas.† I propose here to deduce two striking properties of divisibility sequences which do not depend on the fact that the sequence is linear.

2. *Preliminary Definitions.* An integer m will be said to be a *divisor* of (u) if it divides some term of (u), and a *prime divisor* if it is a prime. The suffix of the first term of (u) divisible by m is called the *rank of apparition* of m. If p is a prime divisor of (u), the rank of apparition of p^a, if it exists, will be denoted by ρ_a.

If we assume that no term of (u) is zero, we can build up from (u) a set of numbers $[n, r]$, the *binomial coefficients belonging to* (u),‡ defined by

$$[n, r] = 1, \qquad (r = 0; n = 0, 1, 2, \cdots),$$

$$[n, r] = u_n \cdot u_{n-1} \cdot \cdots \cdot u_{n-r+1}/u_1 \cdot u_2 \cdot \cdots \cdot u_r, \qquad (r = 1, \cdots, n; n = 1, 2, \cdots).$$

They will not in general be rational integers.

If a and b are any rational integers, we shall write as usual $a | b$ for a divides b and (a, b) for the greatest common divisor of a

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* See Marshall Hall, *Divisibility sequences of the third order*, American Journal of Mathematics, vol. 58 (1936), pp. 577–584, for an account of these sequences and references to the work of Pierce, Poulet, and Lehmer.

† u_n equals the function $(\alpha^n - \beta^n)/(\alpha - \beta)$ up to a constant factor.

‡ For a systematic account of the remarkable properties of these numbers formed from any sequence (u) with no non-vanishing terms see Morgan Ward, *A calculus of sequences*, American Journal of Mathematics, vol. 58 (1936), pp. 255–266.