THE WARING PROBLEM AND ITS GENERALIZATIONS*

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1. *Historical Background.* The past year has witnessed greater advances in the field which is the title of this paper than were made during the whole of its previous long history.

Additive number theory had its origin exactly 300 years ago in Fermat's assertion that he was the first to discover the general theorem that every integer is a sum of three triangular numbers, also is a sum of four squares, also is a sum of five pentagonal numbers, and so on to infinity. Here a triangular number $x(x+1)/2$ counts the number of equal spheres arranged in the form of a triangle with x spheres at the base. The proof which he claimed to possess was never published. Cauchy was the first to publish a complete proof in 1813.

In 1659, Fermat stated that his proof of the 4-square theorem was by descent; namely, any integer is a sum of 4 squares provided a certain smaller integer is such. Recently† there was published such a proof, no detail of which was beyond Fermat. The first published proof was that by Lagrange in 1770.

In the same year, Waring conjectured that every positive integer is a sum of 9 integral cubes, positive or zero, also is a sum of 19 fourth powers, and so on. His assertions were certainly empirical, based on short tables.

About 1772, J. A. Euler (son of the celebrated L. Euler) stated that, in order to express every positive integer as a sum of positive nth powers, at least I terms are necessary, where

$$I = 2^n + q - 2, \qquad q = [(3/2)^n], \tag{1}$$

where, as usual, the notation $[x]$ denotes the largest integer $\leqq x$. He doubtless knew that we must add I powers 1 or 2^n to obtain the sum $2^n q - 1$. For example, we need 4 squares, 9 cubes, 19 fourth powers, 37 fifth powers, and so on.

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† Dickson, American Journal of Mathematics, vol. 46 (1924), p. 2.