ON CERTAIN THEOREMS OF PÓLYA AND BERNSTEIN

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1. *Introduction.* As a generalization of the Fabry gap theorem Pólya† has proved the following result.

THEOREM 1. *The function $f(z)$ given by the power series*

$$f(z) = \sum_{0}^{\infty} a_n z^{m_n} \tag{1}$$

with

$$\lim_{n\to\infty} \frac{n}{m_n} = D, \tag{2}$$

has‡ at least one singularity on every arc on its circle of convergence whose intercepted angle exceeds $2\pi D$.

V. Bernstein§ has proved the following theorem.

THEOREM 2. *Let $\phi(z)$ be analytic in some sector $|\text{am } z| \leqq \alpha$. Let*

$$\overline{\lim_{r\to\infty}} \frac{\log |\phi(r)|}{r} = a. \tag{3}$$

Suppose there exists some b such that for any $\epsilon > 0$,

$$\phi(re^{i\theta}) = O(e^{(a\cos\theta + b|\sin\theta| + \epsilon)r}), \qquad (|\theta| \leqq \alpha). \tag{4}$$

Let $\{\lambda_n\}$ be an increasing sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D, \qquad \lambda_{n+1} - \lambda_n \geqq d > 0. \tag{5}$$

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† Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen,* Mathematische Zeitschrift, vol. 29 (1929).

‡ The theorem with (2) replaced by the Pólya maximal density is obviously an immediate corollary of Theorem 1.

§ Bernstein, *Séries de Dirichlet,* 1933, Chap. 9.