ON THE NON-VANISHING OF THE JACOBIAN IN CERTAIN ONE-TO-ONE MAPPINGS

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THEOREM 1. *If $u(x, y)$ and $v(x, y)$ are harmonic, $u(0, 0) = v(0, 0) = 0$, and if there exists a neighborhood N_1 of the origin of the xy plane and a neighborhood N_2 of the origin of the uv plane such that $u(x, y)$ and $v(x, y)$ establish a mapping of N_1 onto N_2 which is one-to-one both ways, then the Jacobian $\partial(u, v)/\partial(x, y)$ does not vanish at the origin.*

PROOF. As the statement of Theorem 1 remains invariant under homogeneous linear transformations of the uv plane, we may assume, in the developments in polar coordinates for u and v, that

$$u = \sum_{i}^{\infty} [a_n r^n \cos n\theta + b_n r^n \sin n\theta], \qquad (a_i^2 + b_i^2 \neq 0),$$

$$v = \sum_{k}^{\infty} [A_n r^n \cos n\theta + B_n r^n \sin n\theta], \qquad (A_k^2 + B_k^2 \neq 0),$$

that the positive index i does not exceed k, and that for $i=k$ we have $a_k B_k - A_k b_k \neq 0$. Considering the case $i=k$ first, we may, because of the invariance mentioned, assume

$$a_k = B_k = 1, \qquad b_k = A_k = 0.$$

For small values of r, the auxiliary mapping,

$$\bar{u} = r^k \cos k\theta, \qquad \bar{v} = r^k \sin k\theta,$$

can be continuously joined with the given one for $0 \leqq t \leqq 1$ by

$$u_t = r^k \cos k\theta + t\sum_{k+1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

$$v_t = r^k \sin k\theta + t\sum_{k+1}^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta).$$

The vector (u_t, v_t) thereby never differs by more than a vector of length $r^k/2$ from the vector $(\bar{u}, \bar{v})$ whose length is r^k. Thus