A CERTAIN MEAN-VALUE PROBLEM IN STATISTICS*

BY A. T. CRAIG

1. *Introduction.* It is the purpose of this paper to investigate, by means of the characteristic function, the arithmetic mean value, or mathematical expectation, of the sum of the squares of n normally and independently distributed variables when those variables are subject to $m<n$ linear restrictions. For example, if $x_1, x_2, \cdots, x_n$ are n independent values of a variable x which is normally distributed with mean zero and variance σ^2, then the expected value of $\sum_1^n x_j^2$ is $n\sigma^2$. However, the expected value of $\sum_1^n (x_j - \bar{x})^2$, where $n\bar{x} = \sum_1^n x_j$, is $(n-1)\sigma^2$. It is fairly obvious that the latter example could be stated: if the x's are subject to the linear restriction $\sum_1^n x_j = 0$, the expected value of $\sum_1^n x_j^2$ is $(n-1)\sigma^2$. The numbers n and $n-1$, which are equal respectively to the ranks of the matrices of the two quadratic forms, are frequently called the number of degrees of freedom of those quadratic forms.

Let x be subject to the normal law of error

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-x^2/2\sigma^2}$$

and let $x_1, x_2, \cdots, x_n$, be n independent values of x. Write

$$v = \sum_1^n x_j^2, \qquad u_1 = \sum_1^n a_{1j}x_j, \cdots, \qquad u_m = \sum_1^n a_{mj}x_j,$$

in which the a's are real numbers. We wish to find the mathematical expectation of v when $u_1, u_2, \cdots, u_m$ are assigned values which make the system consistent. It is well known that the variables $u_1, u_2, \cdots, u_m$ are normally correlated with variances and covariances given by $\sigma^2 \sum_r a_{jr}a_{kr}$.

2. *The Characteristic Function.* The characteristic function of the joint distribution of $v, u_1, \cdots, u_m$ is

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