ON THE COMPLETENESS OF LAMBERT FUNCTIONS*

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1. *Introduction.* In the theory of Lambert series one encounters the following two sets of functions

$$S_1\colon \frac{(1-t)t^n}{1-t^n}, \quad \text{and} \quad S_2\colon \frac{(1-t)^2t^n}{(1-t^n)^2}, \quad (n=1,2,3,\cdots).$$

Both sets evidently belong to the spaces $L_p(0, 1)$, $p \geqq 1$, and to $C[0, 1]$, the space of functions continuous in the closed interval $[0, 1]$, but nothing definite seems to be known about their closure properties in these various spaces. Wiener's work on the Lambert-Tauber theorem† suggests that they are complete in $L_1(0, 1)$. On the other hand, a direct elementary proof of this fact would form a first step towards a simple proof of his theorem.

In the present note we shall prove completeness in $L_p(0, 1)$, $p \geqq 1$, of sets of functions of the type

$$S(\alpha, \beta, \lambda_n)\colon \qquad (1-t)^\alpha \sum_{m=1}^{\infty} m^\beta t^{m\lambda_n}, \qquad (n=1,2,3,\cdots),$$

under suitable restrictions on the parameters. It will turn out in particular that the sets S_1 and S_2 are complete in any Lebesgue space. The adjunction of $f_0(t)=1$ to $S(\alpha, \beta, \lambda_n)$ leads to sets complete in $C[0, 1]$, but the sets S_1 and S_2 turn out to be border-line cases in the discussion, and we are not able at present to prove that they can be made complete by this device.

The analysis is capable of very considerable extension. Thus we could replace the factor $(1-t)^\alpha$ by more general multipliers. A more interesting situation is encountered if we replace the coefficients m^β by quantities c_m such that $c_1=1$, $|c_m| \leqq Mm^\beta$, $m>1$. Owing to the number theoretic features of the problem, which are introduced by the inversion formula of Möbius, this

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† See Annals of Mathematics, (2), vol. 33 (1932), pp. 1–100, especially pp. 39–43, and his book, *The Fourier Integral*, 1933, pp. 112–124.