THE CONTINUOUS ITERATION OF REAL FUNCTIONS*

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1. *Continuous Iterations.* Let $E(x)$ be a real, continuous, steadily increasing function of x in the range $-\infty < a \leqq x < \infty$ such that

$$E(x) > x, \qquad (x \geqq a), \tag{1}$$

and let $E_1(x) = E(x)$, $E_2(x) = E(E_1(x))$, $\cdots$ denote its successive iterates. In a previous note in this Bulletin, referred to hereafter as Note, one of us† has developed a simple formula for continuously iterating the function $E(x)$. We propose here to determine *all* continuous iterations of $E(x)$ subject to a restriction to be explained presently.

By a continuous iteration of $E(x)$ we shall understand a real function $\Theta_y(x)$ of the two real variables x and y with the following two properties

(i) $$\Theta_0(x) = x, \qquad \Theta_1(x) = E(x), \quad (x \geqq a).$$

(ii) $$\Theta_{y+z}(x) = \Theta_y(\Theta_z(x)), \qquad (x \geqq a,\ y,\ z \geqq 0).$$

The restriction which we shall impose upon the functions $\Theta_y(x)$ is the following:

(iii) *$\Theta_y(a)$ is a steadily increasing continuous function of y in the range* $0 \leqq y \leqq 1$.

2. *Prior Investigations.* The continuous iteration of real functions was discussed in detail by A. A. Bennett.‡ So far as the authors are aware, other investigators have confined their attention to the continuous iteration of analytic functions.§ The functional equation (ii) was first considered by A. Korkine,‖ who

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† Ward, *Note on the iteration of functions of one variable*, this Bulletin, vol. 40 (1934), pp. 688–690.

‡ Annals of Mathematics, (2), vol. 17 (1916), pp. 23–69.

§ See the references in the Note.

‖ Bulletin des Sciences Mathématiques, (2), vol. 6 (1882), part 1, pp. 228-242.