ON THE MAGNITUDE OF THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

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Until very recently all the results of the investigations into the magnitude of the coefficients of the cyclotomic polynomial

$$Q_n(x) = \prod_{\delta | n} (1 - x^{n/\delta})^{\mu(\delta)} \tag{1}$$

tended to show that these coefficients are very small indeed. In fact for $n < 105$ all the coefficients are ± 1, and 0, and for $n < 385$ they do not exceed 2 in absolute value.

In 1883 Migotti* showed that the coefficients of $Q_n(x)$ are all ± 1 or 0 for n a product of two primes, but noted that the coefficient of x^7 in $Q_{105}(x)$ is -2. In 1895 Bang† proved that no coefficient of $Q_n(x)$ for $n = pqr$, $(p < q < r$, odd primes), exceeds $p - 1$.

Nothing further was done on the problem until 1931, when I. Schur gave a very ingenious proof of the following theorem.

SCHUR'S THEOREM. *There exist cyclotomic polynomials with coefficients arbitrarily large in absolute value.*

As this proof has not been published, it is given below.‡

PROOF. Let $n = p_1 p_2 \cdots p_t$, where t is odd and $p_1 < p_2 < \cdots < p_t$ are odd primes such that§ $p_1 + p_2 > p_t$. To prove the theorem it is sufficient to show that the coefficient of x^{p_t} in $Q_n(x)$ is $1 - t$. This can be done by taking $Q_n(x)$ modulo x^{p_t+1}. We then get

$$\begin{aligned} Q_n(x) &\equiv \prod_{i=1}^{t} (1 - x^{p_i})/(1 - x) \\ &\equiv (1 + x + \cdots + x^{p_t - 1})(1 - x^{p_1})(1 - x^{p_2}) \cdots (1 - x^{p_{t-1}}) \\ &\equiv (1 + x + \cdots + x^{p_t - 1})(1 - x^{p_1} - x^{p_2} - \cdots - x^{p_{t-1}}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{mod } x^{p_t+1}). \end{aligned}$$

* Sitzungsberichte, Akademie der Wissenschaften, Wien. (math), (2), vol. 87 (1883), pp. 7–14.

† Nyt Tidsskrift for Mathematik, (B), vol. 6 (1895), pp. 6–12.

‡ This proof is essentially the one given by Schur in a letter to Landau.

§ Such a set of primes exists for every t.