THE FORM $wx+xy+yz+zu$

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1. *Introduction.* In the usual notation,

$$N \equiv N[n = wx + xy + yz + zu; \quad w, x, z, u > 0; \quad y \geqq 0]$$

denotes the number of sets (w, x, y, z, u) of integers, subject to the conditions indicated, satisfying the stated equation in which n is an arbitrary constant integer >0. Let $\zeta_r(n)$ denote the sum of the rth powers of all the divisors of n, so that $\zeta_0(n)$ is the number of divisors. Then

$$N = \zeta_2(n) - n\zeta_0(n). \tag{1}$$

This curious result is the only one of the numerous theorems on quadratic forms stated by Liouville for which (apparently) no proof has been published.*

We shall first show that (1) follows from

$$2N' = \zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n), \tag{2}$$

$$N' \equiv N'[n = wx + xy + yz + zu + ux; \quad w, x, y, z > 0; \quad u \geqq 0],$$

and then prove (2). Another similar result is stated in §5.

2. *Equivalence of* (1) *and* (2). The form in N' may be written

$$yz + (z + x)u + x(w + y);$$

and hence, by the conditions on the variables, $w+y\equiv y'>y$. Thus (2) is equivalent to

$$\begin{aligned}\zeta_2(n)-2n\zeta_0(n)+\zeta_1(n) \\ =2N'[n=yz+(z+x)u+xw; \; x, y, z, w>0; \; u\geqq 0; \; w>y].\end{aligned} \tag{3}$$

Applying the substitution $(xz)(yw)$ to the last we see that (3) holds also when the condition $w>y$ is replaced by $w<y$.

Consider now the remaining possibility, $w=y$. The equation becomes

* J. Liouville, Comptes Rendus, Paris, vol. 62 (1866), p. 714; also, Journal de Mathématiques, (2), vol. 12 (1867), pp. 47–48. Noted in Dickson's *History*, vol. 3, p. 237. Liouville points out why the theorem is unusual.