A NECESSARY CONDITION FOR APPROXIMATION BY RATIONAL FUNCTIONS

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It is the object of the present note to establish the following two theorems; terminology is uniform with that of the writer's recent book on approximation:*

THEOREM 1. *In the extended z plane let R be a region whose boundary is denoted by B. Let every component of B either separate the plane into at least two regions or contain in each of its neighborhoods points of an infinite number of components of B each of which separates the plane into at least two regions. Let the function f(z) be single-valued and analytic in R in the neighborhood of B, and let* $\lim_{z_k \to z_0} f(z_k)$ *exist and be equal to zero whenever the points* z_k *lie interior to R and approach a point* z_0 *of B. Then the function f(z) vanishes identically interior to R in the neighborhood of B.*

THEOREM 2. *Let C be an arbitrary closed point set of the extended plane, and let points* z_k *(not necessarily denumerable) be given exterior to C. A necessary and sufficient condition that a function f(z) single-valued and analytic on C can be uniformly approximated as closely as desired on C by a rational function whose poles lie in the points* z_k *is that f(z) can be extended analytically from C so as to be single-valued and analytic in every point of the plane which is separated by C from the points* z_k. *That is to say, the condition is that there should exist a function which is single-valued and analytic not merely on C but also in every point of the plane separated by C from the points* z_k, *and which coincides with f(z) on C.*

These theorems are slightly more general than the corresponding theorems that are given in the book just mentioned (loc. cit., §1.9, Theorem 15; §1.10, Theorem 16). The present Theorem 2 seems to be the definitive result in its field.

The sufficiency of the condition of Theorem 2 has already

* *Interpolation and Approximation by Rational Functions in the Complex Domain*, Colloquium Publications of this Society, vol. 20, 1935.