THREE THEOREMS ON THE ENVELOPE OF EXTREMALS

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1. *Introduction.* We are concerned with the envelope in the small, more specifically with the conjugate locus in the small. In the large, noteworthy papers have recently been written by Whitehead [**4**] and Myers [**3**], and the reader may also refer to the work of the author [**6**].

In the analytic case in the plane the theory is relatively complete. For a brief account and references see Bolza [**1**], pages 357–369. In 3-space Mason and Bliss [**2**] have treated the envelope in the case where the envelope is ordinary. Hahn [**5**] has reduced the minimum problem in 3-space in the non-parametric form to the study of an analytic function of two variables whose Hessian vanishes at the point in question. This transformation of the problem does not however clear up the difficulties inherent in the envelope theory.

There are three theorems on the envelope which go considerably further than the above theory. Of these theorems the first is a topological characterization of a conjugate point, and has been proved by Morse and Littauer [**7**]. The second theorem is a basic result on the analytic representation of the envelope neighboring one of its points. It is an immediate consequence of two theorems proved by the author, one [**9**] on the order of a conjugate point, and the other, Morse ([**6**], page 235), on the continuation of conjugate points. It is similar to a theorem independently derived from the author's results by Whitehead ([**4**], page 690).

The two preceding theorems refer to the analytic case. The theorem to which most of this paper is devoted is not so restricted. It gives sufficient conditions for a relative minimum in the problems in parametric form when the end points A and B of the given extremal g are conjugate.

2. *The Functional.* Let R be an open region in the space of the variables $(x_1, \cdots, x_m) = (x)$. Let

$$F(x_1, \cdots, x_m, r_1, \cdots, r_m) = F(x, r)$$