CONGRUENCES WITH A COMMON MIDDLE ENVELOPE*

BY MALCOLM FOSTER

1. *Introduction.* Let C and $\bar{C}$ be two rectilinear congruences whose corresponding rays l and $\bar{l}$ are parallel; and let M be the point on the unit sphere S at which the normal is parallel to l and $\bar{l}$. We refer the sphere to any isothermal system and take the linear element in the form $ds^2 = e^{2\lambda}(du^2+dv^2)$.† Relative to the moving trihedral at M, whose x axis is chosen tangent to the curve $v=$const., the coordinates of the points in which l and $\bar{l}$ pierce the xy plane will be denoted by (a, b) and $(\bar{a}, \bar{b})$, respectively. Distances on l and $\bar{l}$ will be measured from these points, and the positive direction will be that which corresponds to the outward-drawn normal at M.

It is the purpose of this note to consider such pairs of congruences as C and $\bar{C}$ when they have a common middle envelope, that is, when the distances to the middle points on l and $\bar{l}$ are equal.

2. *Condition that C and $\bar{C}$ have a Common Middle Envelope.* A necessary and sufficient condition that C and $\bar{C}$ have a common middle envelope is that‡

$$\frac{\dfrac{\partial a}{\partial u} + \dfrac{\partial b}{\partial v} + ar_1 - br + 2\xi}{p_1} = \frac{\dfrac{\partial \bar{a}}{\partial u} + \dfrac{\partial \bar{b}}{\partial v} + \bar{a}r_1 - \bar{b}r + 2\xi}{p_1}.$$

This may be written

$$\frac{\partial}{\partial u}(a-\bar{a}) + \frac{\partial}{\partial v}(b-\bar{b}) + (a-\bar{a})\frac{\partial\lambda}{\partial u} + (b-\bar{b})\frac{\partial\lambda}{\partial v} = 0,$$

which, upon multiplication by e^{λ}, becomes

$$\frac{\partial}{\partial u}\left[e^{\lambda}(a-\bar{a})\right] = -\frac{\partial}{\partial v}\left[e^{\lambda}(b-\bar{b})\right];$$

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† Malcolm Foster, *Rectilinear congruences referred to special surfaces*, Annals of Mathematics, (2), vol. 25 (1923), pp. 159–180.

‡ Foster, loc. cit., p. 163, equation (17).