ON SOME EXTREMAL PROPERTIES OF TRIGONOMETRIC POLYNOMIALS WITH REAL ROOTS

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1. *Introduction.* L. Fejér [1],† O. Szász [2], [3], [4] and E. v. Egerváry [5] have found many interesting extremal properties of non-negative trigonometric polynomials. In particular, Szász [3] has found that for every non-negative trigonometric polynomial of order $\leqq n$ with real coefficients,

$$(1)\quad G_n(\theta) = 1 + \Re\sum_{k=1}^{n} \bar{\gamma}_k e^{ik\theta}, \quad (\gamma_k = \alpha_k + i\beta_k;\ k = 1, 2, \cdots, n),$$

the inequality

$$(2)\qquad |\gamma_k| \leqq 2\cos\frac{\pi}{\left[\frac{n}{k}\right]+2}, \qquad (k = 1, 2, \cdots, n),$$

is valid.‡

The object of this note is to find the *minimum* of the modulus of the first coefficient γ_n, supposing that all roots of $G_n(\theta)$ are real. The first problem of this kind has been considered by Blumenthal [6]; we shall return in §4 to his problem and its generalization.

2. *Equality of Roots of $G_n^*(\theta)$ for Problem* 1. Consider the following problem.

PROBLEM 1. *Find the minimum of the modulus of the first coefficient γ_n of a non-negative trigonometric polynomial*

$$G_n(\theta) = 1 + \Re\sum_{k=1}^{n} \bar{\gamma}_k e^{ik\theta}$$

of order n with real roots.

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† Numbers in brackets refer to the Bibliography at the end.

‡ $\Re z$ means real part of z; $[a]$ means the greatest integer $\leqq a$.